Subjective Mean Variance Preferences Without Expected Utility *

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Abstract

Classical derivations of mean variance preferences have all relied upon the expected utility hypothesis. Some widespread experimental studies have uncovered that the expected utility model tends to be systematically violated in practice. Such findings would lead people to doubt the empirical relevance of the literature and the practical effectiveness of the portfolio selection which employ the mean variance model. In this paper, I postulate a set of axioms, in a setting of subjective uncertainty, and demonstrate that my axiom set implies that the investor assigns subjective probability to events and judges each portfolio solely on the basis of mean and variance of its implied distribution over returns, but does not necessarily rank portfolio according to the expected utility. This subjective mean variance model remains intact with a wide body of observed behavior under uncertainty, which are inconsistent with the hypothesis of expected utility maximization. In addition, the subjective probability is essentially non-normal, which ties together a wide body of empirical observations in finance.

JEL classification: D7; D8.

Keywords: Subjective mean variance utility; Objective mean variance utility; Expected utility model; probabilistic sophistication.

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1 INTRODUCTION

The traditional mean variance model of Markowitz [1952] and Tobin [1958] have long been recognized as the cornerstone of choice theory under uncertainty in economics and finance. Many theoretical models, for instance the capital asset pricing model of Sharpe [1964] and Lintner [1965], builds on the assumption that investors’ ranking over portfolios can be represented solely by the mean and variance of the probability distribution over returns. Its simplicity and elegance have led to widespread use even in industry.

The traditional mean variance model can be justified in at least one of three ways. The first approach assumes normal (See Tobin [1958]) distribution of portfolio return. With this assumption, the mean variance model is consistent with expected utility maximization. The second approach relaxes the distribution assumption. Instead, it assumes expected utility maximization with quadratic utility index (See Tobin [1958]). These two approaches have received considerable scrutiny. The employment of mean variance model in many studies often explicitly assume that the return distributions are normal. However, there is an increasing agreement (See Fama and French [1992]) that at least some return distributions are not normal. The second approach may also not always be plausible. The quadratic utility index reveals increasing absolute risk aversion, which implies that beyond a “satiation” level of return the investor prefers less to more return. This property contradicts frequently observed behavior as shown in Pratt [1964] and Arrow [1971]. As Markowitz [2010] (P.2) wrote:

I never – at any time! – assumed that return distributions are Gaussian (normal)... Nor did I ever assume that the investors utility function is quadratic.

To allow the investor’s preferences being represented by a wide class of utility, for instance Logarithmic utility function, Markowitz [1959], Levy and Markowitz [1979] and many others adopt the quadratic approximations to expected utility by a function of mean and variance over some range of returns. However, the quality of approximation heavily depends on the data set involved. It is very likely that it provides an excellent fit for one case but fails in the other case. Therefore, the employment of approximation approach is not always compelling.

Nevertheless, all the approaches above build upon the expected utility framework. For many years, the expected utility paradigm, which relies on the axiomatic foundation of von Neumann and Morgenstern [1944], Savage [1954], Anscombe and Aumann [1963] and many others, has been acknowledged in economics and finance to be normative appeal. However, especially in recent years, it has been criticized both normatively and descriptively. The examples of its systematic

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1The extensive discussion that the portfolio return distributions are not normal appear in Fama and French [1992] and the references there.
violation in practice, where the uncertainty is represented by means of exogenously specified probabilities, so-called objective uncertainty, are pointed out by Allais [1953], Kahneman and Tversky [1979], Rabin [2000] and many others. Those researchers discovered in various experiments that the choice made by the great majority of subjects violate the expected utility hypothesis. Also, in the real world situations like financial market, the uncertainty rarely represents itself in terms of objective probability, but rather as states of nature. Thus, each combination of portfolio and state leads to a specific return. This form of uncertainty representation is called subjective uncertainty. As Machina and Schmeidler [1992] wrote: “If agents do not maximize expected utility in such well-defined settings of objective uncertainty, it is hard to believe that they will do so in real world settings of subjective uncertainty”.²

Another category of criticism of expected utility model puts doubts on the probabilistic belief formation whenever the relative information is scarce. Knight [1921] is the first to notice the situations where probability models fail and henceforth designs a concept of ambiguity to capture them. Ellsberg [1961] Paradox confirms the concept of ambiguity experimentally and discovers that particular choices made by the majority of subjects violate any probabilistic beliefs, and hence are inconsistent with the expected utility model. On the normative ground, Schmeidler [1989] asserts that the expected utility model is not always appeal due to the fact that “the probability attached to an uncertain event does not reflect heuristic amounts of information that led to the assignment of that probability.”³

Although these findings and criticism stimulate the development of non-expected utility and non-Bayesian models in the past 30 years,⁴ mean variance model continues to serve as pillars, especially in the academic and practical field of finance.⁵ It reveals that both practitioners and academics do realize the descriptive invalidity and normative deficiency of expected utility model, but rather insist on the employment of a mean variance model. It seems crucial that we can identify the situations in which a mean variance model and observed behavioral evidence can coexist without any contradiction.

The purpose of this paper is to investigate how much of the behavioral foundation of mean variance model builds upon the hypothesis of expected utility maximization. Put in an alternative way, I develop a new axiomatic derivation of mean variance model which does not necessarily conform to expected utility hypothesis.

²See Machina [1982, 1987] for the detailed discussion about the expected utility model under objective uncertainty. ³See Gilboa, Postlewaite, and Schmeidler [2012] for extensive discussion on the rationality and probabilistic belief formation. ⁴Indeed, in recent years the alternative models, such as Choquet expected utility model of Schmeidler [1989] and maxmin expected utility model of Gilboa and Schmeidler [1989] have been used in applications ranging from finance to game theory and macroeconomics. I refer to Gilboa [2009] for other models and further applications. ⁵See Chapter 1 of Levy [2012] for elaboration.
Specifically, I adopt a setting of subjective uncertainty. In this environment, the uncertainty facing the investor or individual is represented by a set of states of nature. The objects of choice are portfolios or acts, which assign a return or outcome to each state of nature. Essentially, I prove that a set of simple and intuitive axioms characterizes the investor who (i) possesses a subjective probability distribution over states of nature, (ii) is guided by mean and variance with respect to the induced probability distribution on returns, trading-off between the mean and the variance of returns, and (iii) obeys the monotonicity principle. I will call such an investor a subjective mean variance maximizer.

I first show in Theorem 1 that my mean variance model can account for either decrease in risk or increase in risk. The ability to display the property of deceasing in risk is particularly important since the individual with greater wealth take greater risks are frequently observed. I axiomatize my model in Theorem 2 and show that it is flexible enough to accommodate either quasiconcavity or quasiconvexity, in conjunction with the hypothesis of Mixture Symmetry, Proportion Symmetry and Diversification. In addition, I show that my model includes expected utility model with quadratic utility as a special case. In Theorem 3, Independence axiom along with Proportion Symmetry will characterize the expected utility based mean variance model. Another important special case, linear mean variance model, is axiomatized in Theorem 4 by Translation Invariance.

The subjective mean variance model is a simple but quite powerful model of individual behavior under uncertainty. The most prominent property of this model is that the return distribution is subjective, which is derived from choice behavior, so that is not necessarily normal. The authority of subjective probability distribution comes from three sources. First, as mentioned above, Fama [1965], Officer [1972], Gray and French [1990], Zhou [1993] and others conduct various empirical testing and agree that the empirical return distribution is more peaked and has fatter tails than the normal distribution has. Therefore, exploring the theoretical distribution that fits best the empirical data is an important and popular research field. Ross [1978], Chamberlain [1983] and Owen and Rabinovitch [1983] have shown that the mean variance model is consistent with the expected utility model for elliptical family of distribution which includes normal distribution. Unfortunately, so far no studies (see for instance Levy and Duchin [2004]) could discover a distribution in elliptical family to fit best in most cases. This is not surprising since the elliptical family of distributions has a very limited size and fulfills many restrictions. In contrast, with subjective mean variance maximizing hypothesis, one can virtually assume any theoretical distribution and conduct standard statistic tests to see whether the empirical observations are drawn from this distribution. One can also conduct comparative studies where different distributions are considered with different sets of data. In this sense, the subjective mean variance model is quite robust since it can never be rejected by the statistical invalidity of one return distribution assumption.

Second, the subjective probabilistic belief stems from the fact that investors always incorporate
private information or views into asset pricing. It is well-known that the traditional mean variance model along with global CAPM of Black [1972] require return distribution over all the component of the relevant universe to be homogeneously specified. This unrealistic requirement makes the models have surprisingly little impact on the practical world of investment. In the meanwhile, the Black and Litterman [1992] model, which offers a flexibility to combine the market equilibrium with additional subjective probabilistic view of investor, constructs more stable and better diversified portfolio than traditional mean variance model does, and therefore is widely used in the financial industry. Actually it is more realistic to assume that investors have diverse private information and hold different subjective probabilistic beliefs, which are non-normal in general. For these reasons, subjective mean variance model has a great analytic advantage and apparently serves as a credible representation for the investor.

Third, it is widely accepted that hedge fund returns are not normally distributed (See Agarwal and Naik [2004]). In reality there is an increasing interest in hedge funds that are intentionally designed to have non-normal distributions. Non-normality can have considerable consequences on the uncertainty measurement and performance evaluation of hedge funds. This is due to the fact that standard measures are based on the assumption of normality, and are thus unsuitable for hedge funds. In those cases, subjective mean variance model is sufficient to be a fair approximation to reality.

It is also worthwhile to address the descriptive validity of subjective mean variance model. It is shown that this model can, while expected utility model cannot, generate predictions that are consistent with the typical behavior demonstrated in Allais [1953] paradox. In addition, a number of frequently observed behavior, which violate expected utility model, are jointly consistent with the subjective mean variance model. Thus, the wide use of mean variance model do not necessarily contradict to some widespread experimental critics on the expected utility model.

Not surprising, those empirical and experimental findings have led to the development of numerous generalization of mean variance model without expected utility hypothesis. However, these new theories, including Epstein [1985], Maccheroni, Marinacci, and Rustichini [2006] and Maccheroni, Marinacci, and Ruffino [2013], do not provide the fully axiomatization, so that they stand in need of a behavioral foundation. In contrast, in this paper, a “new” mean variance model is proposed and its behavioral foundation is suggested as well.

My axiomatization result adds to the literature in three ways. First, it presents a derivation of a new mean variance form without probability restrictions which neither assumes nor implies the expected utility. The testable axioms will facilitate the evaluation about whether the subjective

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6For example, Goldman Sachs regularly publishes recommendations for investor allocations based on the Black-Littleman model and has issued reports describing the firms experience using the model.

7I thank Tom Sargent for having drawn my attention to the issue discussed in this paragraph.
The mean variance model is a more accurate model of investors’ choice behavior under uncertainty than traditional models. Second, it provides the empirical implications of the adopted specification, which enable measurement of the parameters on the various data sets and thereby assist the explanation of empirical evidence. Third, my axiomatization provides a useful rhetorical tool for model evaluation. The axioms can be judged according to the similarity and relevance to real investment decision problem, hence catalyze the investor for model selection.

In the following section, I present the subjective uncertainty setting and introduce some notations and concepts. I then provide a formal description of subjective mean variance preferences and discuss their properties. In Section 3, I postulate a set of axioms which sufficiently imply the mean variance model in the subjective uncertainty setting without assuming expected utility maximization. In Section 4, two special cases of my model, expected utility with quadratic index and linear mean variance model, are axiomatized. Section 5 offers an overview of descriptive validity of my model. The Appendix contains all the proof.

2 Setup and Background

2.1 Setting

Our formal setting consists of the following concepts:

Subjective uncertainty is represented by a nonempty set $\Omega$ of states of nature, which can be either finite or infinite. The set of all events is denoted by $\mathcal{E} = 2^{\Omega}$, which consists of all subsets of $\Omega$. Let $X = [m, M] \subset \mathbb{R}$ with $m < 0 < M$ denote the set of outcomes (or returns). Let $\Delta(X)$ denote the set of discrete distributions on $X$. The generic element in $\Delta(X)$ is objective lottery $P$. The objects of choice consist of acts (or portfolios) of the form $f : \Omega \rightarrow \Delta(X)$, which is a simple function mapping states to objective lotteries. The set of all acts is denoted by $\mathcal{F}$. Endow $\mathcal{F}$ with the topology of weak convergence.

Throughout our analysis, the simple act can be expressed in the form

$$f = [P_1 \text{ on } E_1; \ldots ; P_i \text{ on } E_i; \ldots ; P_n \text{ on } E_n]$$

which yield objective lottery $P_i$ in event $E_i$, for some partition $\{E_1, \ldots , E_n\}$ of $\Omega$. For simplicity, we identify each objective lottery $P$ with the constant act $[P \text{ on } \Omega]$. Each objective lottery $P$ can

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8Gilboa, Postlewaite, Samuelson, and Schmeidler [2013] discuss in length on the rhetorical value of axiomatic representations.

9We follow Anscombe and Aumann [1963] setting represented by Machina and Schmeidler [1995]. This setting differs from the original Machina and Schmeidler setting in restricting attention to monetary outcomes. This is because we consider the mean variance model per se.

10This assumption means that we allow both loss and gain in the choices. However, this assumption is not essential to our characterization results.

11A function $f$ is said to be simple if its outcome set $f(\Omega) = \{f(\omega)|\omega \in \Omega\}$ is finite.
be expressed in the form
\[ P = (x_1, p_1; \ldots; x_i, p_i; \ldots; x_n, p_n), \]
which yield outcome \( x_i \) with probability \( p_i \). Whenever no confusion arises, we write \( x \) to stand for \( P = (x, 1) \), the objective lottery yielding \( x \) with certainty. Accordingly each subjective lottery \( f \) can be expressed in the form \( f = [x_1 \text{ on } E_1; \ldots; x_i \text{ on } E_i; \ldots; x_n \text{ on } E_n] \).

We model the individual (or investor) preferences on \( \mathcal{F} \) by a binary relation \( \succsim \). As usual, \( \succ \) and \( \sim \) denote its implied strict preference relation and indifference relation, respectively. The preference relation \( \succsim \) is said to be represented by a preference function \( V: \mathcal{F} \to \mathbb{R} \) if \( f \succsim g \Leftrightarrow V(f) \geq V(g) \).

### 2.2 Subjective Mean Variance Utility

As mentioned above, our goal is to obtain conditions on an individual preference over acts which imply that such individual do represent subjective uncertainty by means of additive probabilities, and do evaluate acts on the basis of their implied probability distributions over outcomes by means of mean and variance. The purpose of this subsection is to formalize this notion of subjective mean variance utility in the absence of expected utility hypothesis.

Given a subjective probability measure \( \mu \) on \( \mathcal{E} \), the probability distribution over outcomes implied by each act \( f = [P_1 \text{ on } E_1; \ldots; P_n \text{ on } E_n] \), denoted by the objective lottery \( P_\mu(f) \), is given by:
\[ P_\mu(f) = \mu(E_1) \cdot P_1 + \cdots + \mu(E_n) \cdot P_n. \]

Given this, we can formalize the notion of mean and variance of act \( f \). For objective lottery \( P \), we denote \( P(x) \) the probability yielding outcome \( x \). The mean of \( f \), denoted by \( E_\mu(f) \) is given by:
\[ E_\mu(f) = \sum_{x \in X} x \cdot P_\mu(f)(x) = \sum_{x \in X} \left( \sum_{i=1}^{n} \mu(E_i)P_i(x) \right) \cdot x. \]

The variance of \( f \), denoted by \( \text{Var}_\mu(f) \), is given by:
\[ \text{Var}_\mu(f) = \sum_{x \in X} (x - E_\mu(f))^2 \cdot P_\mu(f)(x) = \sum_{x \in X} \left[ (x - E_\mu(f))^2 \cdot \sum_{i=1}^{n} \mu(E_i)P_i(x) \right]. \]

The differences between our notion of mean variance model and that of Markowitz are twofold. The first difference, which is fundamental, is that classical mean variance preference of Markowitz satisfy the “Independence Axiom”, whereas subjective mean variance preference need not. The second difference is that we shall remain the monotonicity property, whereas classical mean vari-
The standard notion of monotonicity for preference functionals over probability distributions is monotonicity with respect to first order stochastic dominance. We review them for the case of induced objective lottery $P_\mu(f)$.

Given a probability measure $\mu$ on $\mathcal{E}$, for any two acts $f = [P_1 \text{ on } E_1; \ldots; P_n \text{ on } E_n]$ and $g = [Q_1 \text{ on } F_1; \ldots; Q_m \text{ on } F_m]$, the objective lottery $P_\mu(f)$ is said to \textit{first order stochastically dominate} $P_\mu(g)$ if

$$\sum_{\{y \succ x\}} \left( \sum_{i=1}^{n} \mu(E_i) P_i(y) \right) \geq \sum_{\{y \succ x\}} \left( \sum_{j=1}^{m} \mu(F_j) Q_j(y) \right)$$

for all $x \in X$.

$P_\mu(f)$ \textit{strictly first order stochastically dominates} $P_\mu(g)$ if, in addition, strict inequality holds for some $x \in X$.

Given this, our definition of monotonicity for preference functional over probability distributions is as follows: given a probability measure $\mu$ on $\mathcal{E}$, a preference functional $V$ on $\mathcal{F}$ is said to be \textit{strictly monotonic} if $V(f)(\succ) \geq V(g)$ whenever $P_\mu(f)$ (strictly) first order stochastically dominates $P_\mu(g)$.

Given above, we define a subjective mean variance maximizer as follows:

\textbf{Definition 1.} An individual is said to be a \textit{subjective mean variance} maximizer if there exists a finite-additive probability measure $\mu$ on $\mathcal{E}$, such that their preference relation $\succsim$ over acts can be represented by a strictly monotonic preference functional

$$V(f) \equiv E_\mu(f) + a(E_\mu(f))^2 + b\text{Var}_\mu(f),$$

where $b < 0$.

Two special cases will clarify the scope of the representation. First, if $a = b$, then

$$V(f) = \int (x + ax^2)dP_\mu(f).$$

Thus, expected utility with quadratic utility index is a special case of (1). Second, if $a = 0$, then

$$V(f) = E_\mu(f) + b\text{Var}_\mu(f),$$

This is \textit{linear} mean variance expression, which is most widely used mean variance form.

In the reminder of this subsection we explore some properties of the subjective mean variance utility. Observe first that $V$ satisfy properties such as continuity, strict monotonicity and strict risk
aversion in the sense of aversion to mean preserving spreads. We also frequently observe that individuals with greater wealth take greater risks (see for example Pratt [1964]). In contrast an expected utility maximizer with quadratic index as in (2) will be more averse to constant additive risks about high wealth level than low wealth levels. It is natural to wonder how does the subjective mean variance maximizer’s attitude towards various acts change as his wealth changes.

Given \( P \in \Delta(X) \) and \( x \in X \), we write \( P + x \) to denote the probability distribution that is shifted from \( P \) towards greater terminal outcome positions by \( x \). The question we are asking is, how will individual choices be affected if all the possible outcomes are increased by a constant amount?

**Definition 2.** Let \( \succeq \) be represented by the function \( V \) in (1). \( V \) is said to exhibit *decreasing* (increasing) risk aversion if for all \( P \in \Delta(X), x \in X, w, w' \in X \) such that \( P + w, P + w', w' + x \) all lie in \( \Delta(X) \) and \( w' > w \),

\[
V(P + w) \geq (\leq) V(x + w) \implies V(P + w') \geq (\leq) V(x + w').
\]

\( V \) exhibits *constant risk aversion* if it is both decreasing and increasing risk aversion.

Since each induced objective lottery \( P_{\mu}(f) \) lies in \( \Delta(X) \), our definition is without loss of generality. Intuitively, the definition of decreasing risk aversion says: if individual prefers the risk lottery \( P \) to the sure thing \( x \) at the wealth level \( w \), then he will prefer the sure thing if we increase his wealth level to \( w' \). Since this property is regarded as both normatively and descriptively plausible, we characterize it as following:

**Theorem 1.** Let \( \succeq \) over \( \mathcal{F} \) be represented by function \( V \) in (1). Then \( \succeq \) exhibits the decreasing (increasing) risk aversion if and only if \( a \geq 0 \) (\( a \leq 0 \)).

Thus, \( V \) in (1) exhibits decreasing risk aversion whenever \( V \) is convex in mean. Note that \( V \) defined in (3) is constant risk aversion. Also, the function defined in (2), which is an expected utility function, is increasing in risk aversion.

### 3 Axiom and Main Results

#### 3.1 Axioms

The axioms we need for characterization can be split into three parts. The first part consists of three axioms, namely Weak Order, Continuity and Strict Monotonicity axioms, which sever to imply the existence of a continuous and monotonic preference functional over \( \mathcal{F} \).
Axiom 1. (Weak Order) The relation ≿ on \( F \) is complete and transitive.

Axiom 2. (Continuity) For each \( f \in F \), the sets \( \{ g \in F : f \succsim g \} \) and \( \{ g \in F : g \succsim f \} \) are closed.

Axiom 3. (Strict Monotonicity) For any pair of \( P_i, \hat{P}_i \) in \( \Delta(X) \), if \( P_i \) strictly first order stochastically dominates \( \hat{P}_i \), then

\[
[P_1 \text{ on } E_1; \ldots; P_i \text{ on } E_i; \ldots; P_n \text{ on } E_n] \succ [P_1 \text{ on } E_1; \ldots; \hat{P}_i \text{ on } E_i; \ldots; P_n \text{ on } E_n].
\]

Completeness and transitivity are standard axioms in preferences and do not need further elaboration. Continuity serves as the standard Archimedean property in the context of choice over acts. Axiom 3 is an analogue of the well-known “Substitution Principle” of expected utility (e.g. Savage [1954] and Anscombe and Aumann [1963]), but it is weaker in the sense that it applies in the case where \( P_i \) first order stochastically dominates \( \hat{P}_i \).

The second part consists of six axioms, which exclusively consider the preferences restricted to objective lotteries and suffice to imply mean variance preference functional over objective lotteries. These axioms can be divided into a triple and a pair. The triple corresponds to the axioms of quadratic preferences (e.g. Chew, Epstein, and Segal [1991]), namely Strict Quasiconcavity, or Strict Quasiconvexity and Mixture Symmetry axioms, and serves to imply the existence of a preference functional over objective lotteries which is quadratic in the probabilities.

Axiom 4. (Strict Quasiconcavity) For all \( P, Q \in \Delta(X) \) and all \( \alpha \in (0, 1) \),

\[
P \sim Q \implies \alpha P + (1 - \alpha)Q \succ P.
\]

Axiom 4′. (Strict Quasiconvexity) For all \( P, Q \in \Delta(X) \) and all \( \alpha \in (0, 1) \),

\[
P \sim Q \implies P \succ \alpha P + (1 - \alpha)Q.
\]

Axiom 5. (Mixture Symmetry) For all \( P, Q \in \Delta(X) \) and all \( \alpha \in (0, 1) \),

\[
P \sim Q \implies \alpha P + (1 - \alpha)Q \sim (1 - \alpha)P + \alpha Q.
\]

Axiom 4 states that investors have preferences for hedging (or randomization). Axiom 4′ states that investors have preferences against randomization. Axiom 5 corresponds to the following argument: ‘Suppose you are indifferent between two lotteries \( P \) and \( Q \), and have to choose between an \( \alpha : (1 - \alpha) \) coin flip yielding \( P \) if heads and \( Q \) if tails, or an \( \alpha : (1 - \alpha) \) coin flip yielding \( Q \)
if heads and $P$ if tails. Now either it will land differently, in which cases your choice would not matter, or it will land same, in which cases you are back to a choice between $P$ and $Q$, so you should still be indifferent between both coin flips. Clearly Mixture Symmetry axiom is implied by the standard Independence axiom, but not vice versa.\footnote{We refer to Chew et al. [1991] for further elaboration.}

The pair of axioms, namely \textit{Proportion Symmetry} and \textit{Diversification} axioms, provides the key to our characterization.

\textbf{Axiom 6.} (Proportion Symmetry) For any pair of fair objective lotteries, if $(\frac{1}{2}, x; \frac{1}{2}, y) \sim (\frac{1}{2}, \hat{x}; \frac{1}{2}, \hat{y})$, then for all $\alpha \in (0, 1)$,

$$(\frac{1}{2}, \alpha x + (1 - \alpha)\hat{x}; \frac{1}{2}, \alpha y + (1 - \alpha)\hat{y}) \sim (\frac{1}{2}, (1 - \alpha)x + \alpha\hat{x}; \frac{1}{2}, (1 - \alpha)y + \alpha\hat{y}).$$

This axiom normatively corresponds to the following argument: ‘Say you, an investor, will divide a dollar between two assets, which are two indifferent fair lotteries $(\frac{1}{2}, x; \frac{1}{2}, y)$ and $(\frac{1}{2}, \hat{x}; \frac{1}{2}, \hat{y})$. Now you are asked to choose between a portfolio whose proportion of two assets are $\alpha : (1 - \alpha)$, equivalently lottery $(\frac{1}{2}, \alpha x + (1 - \alpha)\hat{x}; \frac{1}{2}, \alpha y + (1 - \alpha)\hat{y})$, or a portfolio whose proportion of two assets are $(1 - \alpha) : \alpha$, equivalently lottery $(\frac{1}{2}, (1 - \alpha)x + \alpha\hat{x}; \frac{1}{2}, (1 - \alpha)y + \alpha\hat{y})$. Since the original two assets are indifferent, so you should be indifferent between two symmetric combination of the assets’.

It is straightforward to show that a subjective mean variance maximizer should satisfy the proportion symmetry property for general lotteries. But, we only need this property restricted to fair lotteries for our characterization.

\textbf{Axiom 7.} (Diversification) For any pair of fair objective lotteries, if $(\frac{1}{2}, x; \frac{1}{2}, y) \sim (\frac{1}{2}, \hat{x}; \frac{1}{2}, \hat{y})$, then for all $\alpha \in (0, 1)$,

$$(\frac{1}{2}, \alpha x + (1 - \alpha)\hat{x}; \frac{1}{2}, \alpha y + (1 - \alpha)\hat{y}) \succ (\frac{1}{2}, x; \frac{1}{2}, y).$$

Axiom 7 states that the effectiveness of diversification is always positive. Because of diversification, the attractiveness of a particular portfolio when held in proportion of two indifferent assets can go beyond its appeal when it is a sole asset held by an investor. Although the diversification property holds for general lotteries, we only need it for fair lotteries for our purposes.

The final part is the Replacement axiom, first proposed by Machina and Schmeidler [1995], which provides the existence of additive subjective probability over events and implies that the preference functionals over objective lotteries and general acts are consistent:
**Axiom 8.** (Replacement) For any partition \( \{ E_1, \ldots, E_n \} \), if
\[
\begin{bmatrix}
M & \text{on } E_i \\
m & \text{on } E_j \\
m & \text{on } E_k, k \neq i, j
\end{bmatrix}
\sim
\begin{bmatrix}
\alpha M + (1 - \alpha)m & \text{on } E_i \\
\alpha M + (1 - \alpha)m & \text{on } E_j \\
m & \text{on } E_k, k \neq i, j
\end{bmatrix}
\]
for some probability \( \alpha \in [0, 1] \) and pair of events \( E_i \) and \( E_j \), then
\[
\begin{bmatrix}
P_i & \text{on } E_i \\
P_j & \text{on } E_j \\
P_k & \text{on } E_k, k \neq i, j
\end{bmatrix}
\sim
\begin{bmatrix}
\alpha P_i + (1 - \alpha)P_j & \text{on } E_i \\
\alpha P_i + (1 - \alpha)P_j & \text{on } E_j \\
P_k & \text{on } E_k, k \neq i, j
\end{bmatrix}
\]
for all objective lotteries \( P_1, \ldots, P_n \).

Axiom 8 states that the rate at which the individual is willing to substitute subjective uncertainty across the events \( E_i, E_j \) with objective uncertainty \( \alpha \) does not depend on the outcomes.\(^{13}\)

### 3.2 Characterization

In this subsection the preceding axioms and subjective mean variance preferences are related. A characterization of subjective mean variance preferences in the absence of expected utility hypothesis involves taking the eight Axioms. The main result of this paper demonstrate that if preferences satisfy this set of axioms, there exists a subjective probability over states, and individual behaves as if she evaluates acts based on subjective mean variance utility with respect to the subjective probability. This characterization is stated precisely in the following theorem.

**Theorem 2.** The following conditions on a preference relation \( \succsim \) over \( \mathcal{F} \) are equivalent:


**(ii)** There exists a unique finite-additive probability measure \( \mu \) on \( \mathcal{E} \) such that the relation \( \succsim \) over acts can be represented by a strictly monotonic preference functional

\[
V(f) = E_\mu(f) + a(E_\mu(f))^2 + b\text{Var}_\mu(f),
\]

where \( a < b \) (\( a > b \)) and \( b < 0 \).

\(^{13}\)We refer to Machina and Schmeidler [1995] for further elaboration.
Moreover, if there are two subjective mean variance utility $V$ and $\hat{V}$ that represents $\mathcal{F}$, then there exist $\alpha > 0$ and $\beta$ such that $V(f) = \alpha\hat{V}(f) + \beta$ for all $f \in \mathcal{F}$.

Our axiomatic result is sufficiently flexible so that either strict quasiconcavity and strict quasi-convexity can be accustomed to diversification, which is an important assumption for the analysis of portfolio allocation. The separation between diversification and attitudes towards randomization is particularly important on theoretical ground. For example, the attitude towards randomization plays a critical role on the existence of equilibrium in CAPM.

4 PARTICULAR HYPOTHESIS

In this section, we shall discuss and characterize two special cases of subjective mean variance utility: expected utility-based mean variance utility and linear mean variance utility.

4.1 Expected Utility Hypothesis

As mentioned, it is known from Tobin [1958] that an expected utility maximizer would evaluate all probability distributions solely on the basis of means and variances if and only if their utility index (von Neumann-Morgenstern utility function) took the quadratic form. The expected utility based mean variance models come to dominate finance theory on the bases of its elegant axiomatic development and its analytical power. However, to the best of my knowledge, the behavioral assumption of quadratic utility index is deficient. Therefore, we stand in need of full axiomatization. In this subsection, we show that Proportion Symmetry axiom will characterize the utility index to be quadratic.

For completeness, we write the Independence axiom, which is the key behavioral axiom of the expected utility theory.

**Axiom 9.** (Independence) For all $f, g, h \in \mathcal{F}$ and all $\alpha \in (0, 1)$,

$$f \sim g \implies \alpha f + (1 - \alpha)h \succsim \alpha g + (1 - \alpha)h.$$  

**Theorem 3.** The following conditions on a preference relation $\succsim$ over $\mathcal{F}$ are equivalent:

(i) $\succsim$ satisfies the axioms A1 (Weak Order), A2 (Continuity), A3 (Strict Monotonicity), A6 (Proportion Symmetry), A7 (Diversification) and A9 (Independence).

(ii) There exists a unique finite-additive probability measure $\mu$ on $\mathcal{E}$ such that the relation $\succsim$ over
acts can be represented by a strictly monotonic preference functional

\[ V(f) = \int (x + bx^2) dP_\mu(f), \]

where \( b < 0 \).

This set of axioms imply the expected utility with the property of quadratic utility index, which is based on Anscombe-Aumann setting. However, the extension of the “Proportion Symmetry” of expected utility theory under Savage setting is straightforward.\(^{14}\)

4.2 Constant Risk Aversion

The most widely used mean variance utility functional form is probably the linear mean variance utility function, in which the investor linearly trade-off between mean and variance of returns. Traditionally, such expression is derived from assumptions of an exponential utility index and normal probability distributions. As mentioned, Fama and French [1992] stuck the strongest blow to the normal distribution assumption. They empirically displayed that stock returns are not normal. In response to these argument, we shall show in this subsection that along with a new axiom, we can obtain a characterization of a linear mean variance utility without distribution restriction.

**Axiom 10.** (Translation Invariance) For all objective lotteries \( P_1, P_2, Q_1 \) and \( Q_2 \) in \( \Delta(X) \) such that \( P_1 = Q_1 + w \) and \( P_2 = Q_2 + w \) for some \( w \in \mathbb{R} \),

\[ P_1 \sim P_2 \iff Q_1 \sim Q_2. \]

This axiom gets its name because it states that the individual’s revealed ranking of a pair of lotteries is stable in the sense that it does not depend on the changes of wealth. It is straightforward to see that the preference functional as in (3) has to satisfy this axiom.

By removing the Diversification axiom from our characterization in Theorem 2 and replacing it by Translation Invariance, we obtain the following characterization of linear mean variance utility in the absence of distribution restrictions.

**Theorem 4.** The following conditions on a preference relation \( \succsim \) over \( \mathcal{F} \) are equivalent:

(i) \( \succsim \) satisfies the axioms A1 (Weak Order), A2 (Continuity), A3 (Strict Monotonicity), A4’ (Strict Quasiconvexity), A5 (Mixture Symmetry), A6 (Proportion Symmetry), A8 (Replacement) and A10 (Translation Invariance).

---

\(^{14}\)Such extension can be formulated as: For an event \( E \), if \( xEy \sim \hat{x}E\hat{y} \), then for any \( \alpha \in (0, 1) \), \( (\alpha x + (1 - \alpha)\hat{x})E(\alpha y + (1 - \alpha)\hat{y}) \sim ((1 - \alpha)x + \alpha\hat{x})E((1 - \alpha)y + \alpha\hat{y}). \)
There exists a unique finite-additive probability measure $\mu$ on $\mathcal{E}$ such that the relation $\succeq$ over acts can be represented by a strictly monotonic preference functional

$$V(f) = E_\mu(f) + b\text{Var}_\mu(f)$$

where $b < 0$.

5 Conclusion

In Anscombe-Aumann setting, this paper derives a subjective mean variance utility from choice behavior over acts, which include both objective and subjective lotteries. This model neither assumes nor implies the Independence Axiom, therefore do not necessarily conform to expected utility hypothesis. We argue that other than A1, 2, 3, 4 (4'), 5 and 8, which are standard behavioral assumptions, Translation Invariance and Diversification Symmetry Axioms characterize the subjective mean variance utility.

We close the conclusion by discussing the behavioral validity of subjective mean variance utility. We consider several observations concerning individual choices, and briefly state the invalidity of expected utility model and explain how our axioms as well as subjective mean variance model can be used to account for those types of behavior.

One of best-know objection to expected utility hypothesis is the ‘Allais Paradox’ (Allais [1953]). Here we adopt the version of Machina and Schmeidler [1992], which converted from the usual probability distribution format into acts.

Table 1: Allais Paradox

<table>
<thead>
<tr>
<th></th>
<th># 1</th>
<th># 2 –# 11</th>
<th># 12 – # 100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_1$</td>
<td>$1M$</td>
<td>$1M$</td>
<td>$1M$</td>
</tr>
<tr>
<td>$f_2$</td>
<td>$0$</td>
<td>$5M$</td>
<td>$1M$</td>
</tr>
<tr>
<td>$f_3$</td>
<td>$1M$</td>
<td>$1M$</td>
<td>$0$</td>
</tr>
<tr>
<td>$f_4$</td>
<td>$0$</td>
<td>$5M$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

There is an urn containing 100 numbered balls. A ball is randomly drawn from this urn. An act, for example $f_2$, therefore yields $0$ if the drawn number is #1, $5M$ if the drawn number is between #2 and #11, and $1M$ if the drawn number is between #12 and #100, which is equivalent to a bet with 0.01 : 0.10 : 0.89 chance of winning $0$ or $5M$ or $1M$. (Here $1M$ indicates a million.) Experimental evidence show that the great majority of subjects rank $f_1 \succ f_2$ and $f_4 \succ f_3$. Note that independence axiom implies that $f_1 \succeq f_2$ if and only if $f_3 \succeq f_4$. This is due
to the property of separability between mutually exclusive events. However, Mixture Symmetry places no restrictions on the preferences over the acts \( \{ f_1, f_2 \} \) versus \( \{ f_3, f_4 \} \), which is why it can characterize subjective mean variance model as well as does not conform to expected utility hypothesis. Below we take a numerical calculation, which display that a subjective mean variance model, in which \( \theta = 10^{-6} \), can adapt the Allais-type behavior in this example.

\[
V(f_1) = 1000000 > V(f_2) = 1390000 - 10^{-6} \times (1.4579 \times 10^{12}) = -67900
\]

\[
V(f_3) = 110000 - 10^{-6} \times (9.79 \times 10^{10}) = 12100 <
\]

\[
V(f_4) = 5 \times 10^6 - 10^{-6} \times (2.25 \times 10^{12}) = 2.75 \times 10^6
\]

However, this present model cannot claim to have come close to exhausting the empirical evidence. It is clear that our model is contradicted by the typical behavior in the Ellsberg Paradox. This is to be expected since such behavior is simply inconsistent with the existence of unique subjective probability. There is clearly room for more work to extend the analysis from the case of unique subjective probability to multiple probabilities, which would allow for a corresponding ‘aversion to ambiguity’ theory of mean variance utility. In addition, it would seem appropriate to distinguish two sources, namely risk and ambiguity, which would lead to a different representation for objective and subjective lotteries.\(^{15}\)

\[
\text{Acknowledgement}
\]

TBA.

\section*{APPENDIX: PROOF OF THEOREM}

\section{A PROOF OF THEOREM 1}

For \( p \in \Delta(X) \), we write \( \mu \) and \( \sigma^2 \) to denote the mean and variance of \( P \). It suffice to show that if \( V(P + w) = V(x + w) \) for some \( x, w \in X \), then for all \( w' > w \), \( V(P + w') \geq V(x + w') \) if and only if \( a \geq 0 \).

\( V(P + w) = V(x + w) \) implies

\[
(\mu - x) + a(\mu - x)(\mu + x + 2w) + b\sigma^2 = 0
\]

\(^{15}\)See Chew and Sagi [2008] and Ergin and Gul [2009], for corresponding analyses of the sources, and Strzalecki [2011] for an analysis on Anscombe and Aumann framework.
Let $\delta = w' - w > 0$. Then,

$$V(P + w') - V(x + w')$$

$$= (\mu - x) + a(\mu - x)(\mu + x + 2w) + 2a\delta(\mu - x) + b\sigma^2$$

$$= 2a\delta(\mu - x)$$

Since $\mu > x$, $V(P + w') - V(x + w') \geq 0$ iff $a \geq 0$.

**B Proofs of Theorem 2**

The implication $(ii) \Rightarrow (i)$ is straightforward. We only provide the proof $(i) \Rightarrow (ii)$. Consider the preference relation $\succsim$ over $\mathcal{P}$ satisfying Axioms 1, 2, 3, 4, 5, 6, 7, and 8. (A similar argument applies to Axiom 4' instead of Axiom 4.) Our proof consists of three steps. Step 1 displays that subjective mean variance preferences over objective lotteries belongs to a class of preferences whose representation functions are quadratic in probabilities. Also, we show that preference over objective lotteries can be represented by a function that is quadratic in probabilities. Step 2 shows that preference over objective lotteries can be represented by a subjective mean variance utility function as in (1). Finally, Step 3 derives a unique subjective probability measure $\mu$ and show that individual is indifferent between a general act $f$ and its induced objective lottery $P_{\mu}(f)$. Therefore, we can represent preference over $\mathcal{P}$ by $V$ in (1).

**Step 1.** Recall that a preference function $V$ over $\Delta(X)$ is said to be quadratic in probabilities if it has the following expression:

$$V(P) = \iint \phi(x, y) dP(x) dP(y),$$

for some symmetric function $\phi : X \times X \rightarrow \mathbb{R}$. Note that our mean variance representation in (1) corresponding to

$$\phi(x, y) = \frac{1}{2}(x + y) + (a - b)xy + \frac{b}{2}(x^2 + y^2)$$

According to Chew et al. [1991] (Theorem 5, P.149), Weak Order, Continuity, Monotonicity, Strict Quasiconcavity and Mixture Symmetry imply that there exists a function $V$ as in (4) represents $\succsim$ on $\Delta(X)$. It is clear that $\phi$ inherits the properties of continuity and monotonicity from $\succsim$. 

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Step 2. Consider $\succeq$ over the set

$$\mathcal{P} = \{ P = (\frac{1}{2}, x; \frac{1}{2}, y) : x, y \in X \}. $$

$\mathcal{P} \subseteq \Delta(X)$ is the set of binary objective lotteries with equal probabilities. It is straightforward to verify that

$$V(\frac{1}{2}, x; \frac{1}{2}, y) \equiv \tilde{V}(x, y) = \frac{1}{4} (\phi(x, x) + \phi(y, y) + 2\phi(x, y))$$

represents $\succeq$ over $\mathcal{P}$.

Since $\succeq$ over $\mathcal{P}$ satisfies the Proportion Symmetry and Diversification axioms, we know from Chew et al. [1991] (App 3, P. 155) that $\tilde{V}$ is a quadratic function on $X^2$. By symmetry, function $\tilde{V}$ has the following expression:

$$\tilde{V}(x, y) = \alpha (x + y) + \beta xy + \gamma (x^2 + y^2).$$

By Strict Monotonicity, $\alpha > 0$. According to Diversification, if $\tilde{V}(x, y) = \tilde{V}(\hat{x}, \hat{y})$, then

$$\tilde{V}(\frac{x + \hat{x}}{2}, \frac{y + \hat{y}}{2}) > \tilde{V}(x, y).$$

This implies

$$-\beta(x - \hat{x})(y - \hat{y}) - \gamma [(x - \hat{x})^2 + (y - \hat{y})^2] > 0.$$ 

Therefore, we must have $2\gamma - \beta < 0$.

We now renormalize it by letting $\alpha = \frac{1}{2}, \beta = \frac{a - b}{2}$ and $\gamma = \frac{a + b}{4}$. It is well-defined since for each pair of $\beta$ and $\gamma$, there is a unique pair of $a$ and $b$ satisfying those. Hence,

$$\tilde{V}(x, y) = \frac{x + y}{2} + \frac{a - b}{2} xy + \frac{a + b}{4} (x^2 + y^2),$$

where $a = 2\gamma + \beta$ and $b = 2\gamma - \beta < 0$. Together with Equation 6, the expression of $\phi(x, y)$ is as in Equation 5. Thus, for each $P \in \Delta(X)$,

$$V(P) = E(P) + a(E(P))^2 + bVar(P).$$

We are left to show that $a < b$. In addition, $\succeq$ satisfies Strict Quasiconcavity. If $V(P) = V(Q)$,
then $V(\alpha P + (1 - \alpha)Q) > V(P)$ for any $\alpha \in (0, 1)$. This implies that

$$\alpha \int x \, dP + (1 - \alpha) \int x \, dQ + (a - b)[\alpha \int x \, dP + (1 - \alpha) \int x \, dQ]^2 + b[\alpha \int x^2 \, dP + (1 - \alpha) \int x^2 \, dQ]$$

$$- (\int x \, dP + (a - b)(\int x \, dP)^2 + b \int x^2 \, dP) > 0$$

This is equivalent to

$$-(a - b)\alpha(1 - \alpha)(\int x \, dP - \int x \, dQ)^2 > 0.$$  

Hence, $a < b$.

**Step 3.** Since $\succeq$ over $\mathcal{F}$ satisfies Weak Order, Continuity, Strict Monotonicity and Replacement axioms, Machina-Schmeidler Theorem (Machina and Schmeidler [1995], P. 119) implies that there exists a unique finitely-additive probability $\mu$ on $\mathcal{E}$ such that $V(f) = V(P\mu(f))$.

To see the uniqueness, assume that $V$ and $V'$ are subjective mean variance utility defined on $\mathcal{F}$ such that $V = \psi(V')$ for some increasing function $\psi$. Since both $V$ and $V'$ are quadratic functions, by computing the Hessians of $V$ and $\psi(V')$ it is straightforward to see that $\phi$ must be linear.

**C PROOF OF THEOREM 3**

The implication $(ii) \Rightarrow (i)$ is straightforward. We only show the proof $(i) \Rightarrow (ii)$. Consider the preference relation $\succeq$ over $\mathcal{F}$ satisfying Axioms 1, 2, 3, 6, 7 and 9. Clearly, Axioms 1, 2, 3 and 9 implies that there exists a unique finitely-additive probability measure $\mu$ on $\mathcal{E}$ and a utility function $u$ on $X$ such that for all $f \in \mathcal{F}$,

$$V(f) = \int u(x) \, dP\mu(f)$$

represents $\succeq$ over $\mathcal{F}$. We need to show that $u$ is a quadratic function.

Consider again $\succeq$ over the set $\mathcal{P}$. Proportion Symmetry and Diversification imply that $V$ is quadratic restricted to $\mathcal{P}$. Hence, from the Step 2 of Proof of Theorem 2,

$$\frac{1}{2}(u(x) + u(y)) = \frac{x + y}{2} + \frac{a - b}{2}xy + \frac{a + b}{4}(x^2 + y^2).$$

Clearly, $a = b$. Hence $u(x) = x + \frac{b}{2}x^2$, where $b < 0$. 

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The implication \((\text{ii}) \Rightarrow (\text{i})\) is straightforward. We only provide the proof \((\text{i}) \Rightarrow (\text{ii})\). Consider the preference relation \(\succsim\) over \(\mathcal{F}\) satisfying Axioms 1, 2, 3, 4’, 5, 6, 8 and 10. According to the proof of Theorem 2, there is a representation function \(V\) on \(\mathcal{F}\) which is quadratic on probabilities as in Equation 4. It suffices to show that the function \(\phi\) associated with \(V\) has the following expression:

\[
\phi(x, y) = \frac{x + y}{2} + b\frac{(x - y)^2}{2},
\]

where \(b < 0\). Note that we do not assume Diversification axiom. Therefore, we could not claim directly that \(\phi\) is a quadratic function.

Consider again the preference relation \(\succsim\) over \(\mathcal{P}\). We want to show that \(\succsim\) on \(\mathcal{P}\) satisfies either Diversification or Aversion to Diversification, i.e. if \((\frac{1}{2}, x; \frac{1}{2}, y) \sim (\frac{1}{2}, \hat{x}; \frac{1}{2}, \hat{y})\), then for all \(\alpha \in (0, 1)\), \((\frac{1}{2}, x; \frac{1}{2}, y) \succ (\frac{1}{2}, \alpha x + (1 - \alpha)\hat{x}; \frac{1}{2}, \alpha y + (1 - \alpha)\hat{y})\).

As we already known from Chew et al. [1991] (App 1, p. 154), each indifference curve in \(\mathcal{P}\) is either strictly convex, or strictly concave or linear. If one indifference curve is strictly convex (or strictly concave, or linear), then Translation Invariance axiom implies that every indifference curves are strictly convex (or strictly concave or linear). Suppose that one indifference curve is linear. Then function \(\phi(x, y)\) must be linear, which means that \(V\) has an expected value expression. However, this contradicts the Strictly Quasiconvexity axiom. Hence, \(\succsim\) satisfies either Diversification or Aversion to Diversification.

Since \(\succsim\) over \(\mathcal{P}\) satisfies either Diversification or Aversion to Diversification, we must have

\[
V(\frac{1}{2}, x; \frac{1}{2}, y) = \tilde{V}(x, y) = \frac{1}{4}\left[\phi(x, x) + \phi(y, y) + 2\phi(x, y)\right]
\]

to be quadratic. Hence

\[
\tilde{V}(x, y) = \alpha(x + y) + \beta xy + \gamma(x^2 + y^2).
\]

By Monotonicity, \(\alpha > 0\). Now suppose that \(\tilde{V}(x, y) = \tilde{V}(\hat{x}, \hat{y})\). Select a proper \(\delta > 0\) such that \(x + \delta, y + \delta, \hat{x} + \delta, \hat{y} + \delta \in X\). According to Translation Invariance,

\[
\tilde{V}(x + \delta, y + \delta) - \tilde{V}(\hat{x} + \delta, \hat{y} + \delta)
\]

\[
= \alpha(x + y + 2\delta) + \beta(x + \delta)(y + \delta) + \gamma((x + \delta)^2 + (y + \delta)^2)
\]

\[
- \alpha(\hat{x} + \hat{y} + 2\delta) - \beta(\hat{x} + \delta)(\hat{y} + \delta) + \gamma((\hat{x} + \delta)^2 + (\hat{y} + \delta)^2)
\]

\[
= \beta\delta(x + y - \hat{x} - \hat{y}) + 2\gamma\delta(x + y - \hat{x} - \hat{y})
\]

\[
= 0
\]
This implies that $\beta = -2\gamma$. We can renormalize it by letting $\alpha = \frac{1}{2}$ and $\gamma = \frac{b}{4}$. So,

$$\tilde{V}(x,y) = \frac{x + y}{2} + \frac{b}{4}(x - y)^2.$$  

Hence, $\phi(x,y)$ has the expression as in Equation 7. We are left to show that $b < 0$. Strict Quasi-convexity implies that if $V(P) = V(Q)$, then for any $\alpha \in (0, 1)$

$$V(P) - V(\alpha P + (1 - \alpha)Q)$$

$$= \int xdP + b(\int x^2dP - \int xdP)^2 - \alpha \int xdP - (1 - \alpha) \int xdQ$$

$$+ b[\alpha \int xdP + (1 - \alpha) \int jdQ] - \alpha \int x^2dP + (1 - \alpha) \int x^2dQ]$$

$$= -b\alpha(1 - \alpha)(\int xdP - \int xdQ)^2$$

$$> 0$$

Hence $b < 0$.

REFERENCES


