Utilitarian Aggregation with Reasonably Heterogeneous Beliefs*

Antoine Billot† and Xiangyu Qu‡

September 2017

Abstract

The utilitarian aggregation rule requires social utility and beliefs to be respectively the affine aggregation of individual utilities and beliefs. Since, in case of belief heterogeneity, the standard Pareto condition is incompatible with such a separate aggregation, a new condition, called the *belief-proof Pareto condition*, is proposed to alleviate occurrences of spurious agreement by restricting unanimity to beliefs that can be considered 'reasonable' by society. Then, we show, in the Anscombe-Aumann framework (Theorems 1 and 2) as well as in Savage's (Theorems 3 and 4), that the belief-proof Pareto condition is equivalent to separate aggregation under complete or incomplete information of society on individual beliefs.

1 Introduction

1.1 Outline

Harsanyi [1955] proposes an axiomatic justification of the utilitarian aggregation rule that is based, on the one hand, on Bayesian rationality and, on the other hand, on the Pareto condition. These two principles are well known to imply social utility to be a combination of individual utilities (Harsanyi's Aggregation Theorem). Nevertheless, in a framework *à la* Savage [1954] where individual beliefs are subjective and thus possibly heterogeneous, these two requirements are rather

*This research has been conducted as part of the project Labex MMEDII (ANR 11-LABX-0033-01).

†LEMMA, université Panthéon-Assas (Paris 2). Email: billot@u-paris2.fr

‡CNRS, Centre d'économie de la Sorbonne. Email: xiangyuqu@gmail.com

problematic. Hylland and Zeckhauser [1979] and Mongin [1995, 2015] prove that a society wishing to aggregate individual preferences into social ones faces an impossibility: in case of belief heterogeneity, either Bayesian rationality, *i.e.*, subjective expected utility, as imposed on society, or the standard Pareto condition should be rejected (see, Mongin and Pivato [2015] for a general discussion of this impossibility result). Unfortunately, many important social decisions involve an uncertain environment where both individual tastes and beliefs are heterogeneous.

When facing such a substantial impossibility, it is recognized that welfare economics is unable to reach a true consensus in relation to the evaluation of social options under subjective uncertainty. At least two main approaches have been proposed to deal with this impossibility. On the one hand, the *ambiguity approach* calls for retaining the original Pareto condition in its full strength and relaxing the assumption of social preferences consistent with the *subjective* (or Savage) *expected utility model* (SEU).[1] On the other hand, the *Savagian approach* advised by Mongin [1995, 2015] calls for retaining the assumption of SEU social preferences and, consequently, criticizes the Pareto condition as a normative principle.

However, at first glance, the Pareto condition seems to be uncontroversial. How indeed could people oppose a society choosing one option against another if every individual prefers the former to the latter? However, Gilboa, Samet, and Schmeidler [2004] (GSS) produce an example—a duel—in which both agents prefer fighting to not fighting where this agreement is based on conflicting beliefs. Such a unanimity without unanimous reasons—so-called *spurious unanimity* by Mongin [1995]—has indeed no true normative standing. It does not translate the existence of an agreement between individuals. Consequently, none of them can expect to genuinely increase his welfare. Hence, the standard Pareto condition cannot be considered compelling in case of spurious unanimity. But then, what could be a defensible and plausible version of this condition that would be immune to spurious unanimity?

GSS propose an elegant and attractive but demanding solution corresponding to a weakening of the Pareto condition according to which preferences unanimity can only be considered conclusive if unanimous preferences are based on agreed-upon beliefs.[2] Actually, the basic impossibility in case of belief heterogeneity reveals the existence of inadmissible profiles of beliefs and tastes, especially when a spurious unanimity takes shelter behind an apparent unanimity as in GSS's duel example. Hence, under a Bayesian rationality, the only way to modify the Pareto condition is to circumvent these inadmissible profiles in order to take into account the spurious unanimity

[1]For instance, Crès, Gilboa, and Vieille [2011], Alon and Gayer [2016] and Qu [2017] propose that social preferences admit a *maxmin expected utility* (MEU) representation while Danan, Gajdos, Hill, and Tallon [2016] propose that social preferences are of the Bewley type.

[2]However, for being applied within SEU, this suggestion needs individual priors to conform to countable additivity.

objection to the *ex ante* Pareto condition. Recent versions of the Pareto condition are thus all restricted to specific collection of acts: GSS restrict the Pareto dominance to common-belief acts, Danan et al. [2016] restrict it to common-taste acts. In the different settings, Gilboa, Samuelson, and Schmeidler [2014] introduce a No-Betting-Pareto condition requiring the existence of shared beliefs on top of unanimity preferences, and Gayer, Gilboa, Samuelson, and Schmeidler [2014] and Brunnermeier, Simsek, and Xiong [2014] adopt the similar kind of restriction, compelling society to accept any unanimity that concerns acts for which one component is agreed-upon across individuals.

The purpose of this paper is to suggest a new criterion, according to which society should regard a prior as *reasonable* only if it is an individual belief. Actually, in our view, a society that cannot or does not want to discriminate among individuals is basically led to adopt a somehow 'electoral' principle: one man, one prior. Consequently, society is here supposed to restrict unanimity solely to cases where this principle can be active, that is whenever individual preferences remain unchanged with respect to any other individual prior. Formally, we assume that both individuals and society have expected utility preferences and we single out a new Pareto condition, called the *belief-proof Pareto condition*, that is supposed to alleviate spurious unanimities. Then, this condition being sufficiently general to be applied in broader uncertain environments, we show that, in the Anscombe-Aumann expected utility model (AAEU) and in the Savage model (SEU), the belief-proof Pareto condition implies social utility to be an affine aggregation of individual utilities and social beliefs to be an affine aggregation of individual beliefs as well. In addition, the belief-proof Pareto condition brings us to distinguish two cases in point: in the first case, society has complete information on individual beliefs and, in the second one, it has incomplete information. Of course, most of the contributions in the field of aggregation involve individual data—preferences and beliefs—that are commonly known. However, recently, Nehring [2004] and Chambers and Hayashi [2014] developed an aggregation theory under incomplete information where individuals are assumed to possess private information. For our part, the belief-proof Pareto condition is shown to be sufficiently effective to resolve the impossibility of separate aggregation under both environments.

It is useful to address the essential difference between our *belief-proof Pareto condition* (BPPC) and GSS's *restricted Pareto condition* (RPC). Technically, it is well-known that RPC is equivalent to separate aggregation only if there exist identical events for all possible probabilities. However, assuming the existence of such events is very demanding and, in most cases, impractical. In Anscombe-Aumann's setting and Savage's one, RPC is thus basically too weak to imply even the affine aggregation of individual utilities. BPPC, though stronger than RPC, does not rely on the existence of identical events and, therefore, is sufficient to imply separate aggregation. More deeply, the two conditions envisage social choice situations under which the maximum 'level of heterogeneity' consistent with the utilitarian aggregation is not the same. For GSS, individuals must share the same probability estimates for at least some events to give any hope to the utilitarian aggregation rule: in other words, individuals should achieve a minimum prior consensus which decreases accordingly the level of their heterogeneity. Instead, BPPC does not require such a local prior consensus. Consequently, it allows more prior heterogeneity. We refer to Section 5 for detailed discussion.

In terms of political and social implications, these results seem promising. As a matter of fact, for most important decisions a society has to take, it is quite implausible that there exist identical individual beliefs over at least some events. In addition, all the conditions introduced above do not really depict a genuine process of formation of social beliefs: they rather enforce logical bounds for aggregation of individual beliefs. Consequently, any political representative of a society—for instance a President—might be tempted to achieve beliefs that are unfettered by individual opinions. This of course contradicts the moral principles of democracy but it also possibly endanger social efficiency and political stability. By contrast, the condition we propose (according to the theoretical results) suggests a separate aggregation rule that is respectful of individuals and individual opinions and, obviously, provides a more democratic answer to the issue of the genesis of social decisions.

1.2 Motivation

Let us revisit the famous GSS's example of a duel, showing that the standard Pareto condition—requiring that if all individuals agree on preferences between two alternatives, so should society—may not be very plausible for individuals with heterogeneous priors. Suppose society $\mathcal{I}$ to consist of two duelists, 1 and 2, equipped with SEU preferences u_1 and u_2. Assume the state space Ω to be defined by the two following states $\{1 \text{ wins}, 2 \text{ wins}\}$ and the outcome set X by $\{1$ is injured, 2 is injured, nonduel$\}$. Suppose further that individual utilities and beliefs are defined as follows:

$u_{1,2} \diagdown X$	$x = 2$ is injured	$y = 1$ is injured	$z =$ nonduel
u_1	1	-5	0
u_2	-5	1	0

and

$\pi_{1,2}\diagdown\Omega$	$E = 1$ wins the duel	$F = 2$ wins the duel
π_1	0.9	0.1
π_2	0.1	0.9

Compare now the *duel act* $f = xEy$ with the constant *non-duel act* z. Since both individual preferences are assumed to be SEU, it is straightforward that both individuals prefer dueling to not dueling. Should the society necessarily consider that it is compelling to adopt such a preference, that is to prefer the duel? Like so many others, especially GSS, we think not necessarily. Observe that this apparent unanimity is based on conflicting individual utilities and beliefs. Actually, no single probability could be found such that both individuals prefer the duel to the non-duel. Even though society would believe *ex ante* that there exists a 'reasonable' prior that supports the duel, yet, in this example, no hypothetical prior can rationalize such an apparent unanimity. This unanimity is therefore spurious and indeed it is not appealing for society to comply with it.

Thus, an important question is the following: what are the persuasive reasons for society to comply with unanimity preferences? One may suggest that if there exists a probability distribution based on which unanimity preferences can be rationalized, then it should be compelling for society to adopt these preferences. However, the choice of an arbitrary probability violates methodological individualism—since methodological individualism requires society to be essentially neutral and respectful of individual motives.[3] According to this principle, it is necessary for society to consider *reasonable* any individual beliefs and to accommodate all individual beliefs rather than picking at random any discretionary probability. Consequently, the set of individual beliefs constitutes the breeding ground for all reasonable priors. Hence, since, on the one hand, any individual prior is reasonable for society and, on the other hand, to be compelling, unanimity preferences are expected to preserve unanimity under reasonable beliefs, unanimity preferences should therefore preserve unanimity under any individual prior. Actually, this is consistent with the modern postulate (*à la* Adam Przeworski, for instance) whereby a democratic society should commit itself not to discriminate among individuals. Any society aiming to aggregate opinions and upholding some fairness requirement is basically led to adopt a somehow 'electoral' principle: one man, one prior. Consequently, society is here supposed to restrict unanimity solely to cases where this principle can be active, that is whenever individual preferences remain unchanged with respect to any individual prior.

In fact, in the original GSS's duel example, unanimity is not always preserved, that is for both

[3]We refer to Hayek [1948] for a detailed discussion of the epistemological necessity of individualism to study social phenomenons and Arrow [1994] for a modern discussion.

priors. Consider now a slightly modified duel example where individual utilities and beliefs are given as follows:

$\pi_{1,2}\diagdown\Omega$	E	F
π_1	0.9	0.1
π_2	0.1	0.9

and

$u_{1,2}\diagdown X$	x	y	z
u_1	10	-1	0
u_2	-1	10	0

The two duelists improve their respective utilities for x and y, while they maintain the ranking among x, y and z. Both individuals still prefer the duel act $f = xEy$ to the nonduel act z. However, this unanimous preference can now be supported by any 'reasonable' prior, *i.e.*, π_1 and π_2: if we switch individual beliefs (from π_1 to π_2 for 1 and vice versa for 2), both individual preferences among f and z remain unchanged—and unanimous. Such a unanimity is clearly independent of individual beliefs in the sense where all permutations between priors do not modify individual preferences. Then, it seems sensible for society to adopt it.[4]

This situation is reminiscent of a strategy-proof game, namely a game that is immune to information manipulation. By analogy, we called *belief-proof Pareto* a condition such that, every time unanimity preferences can be rationalized by all individual priors, society complies with it—*i.e.*, society is immune to belief manipulation. In contrast to the standard Pareto condition where unanimity is required with respect to individual rationality, the belief-proof Pareto condition rather requires a kind of social rationality in the sense that none of the individual priors prevents unanimity preferences. To a certain extent, in our view, it constitutes a minimal democratic requirement for a society made of individuals with heterogeneous beliefs.

The paper proceeds as follows: in Section 2, the normative background is described and the belief-proof Pareto condition is displayed. Then, we prove (under a mild assumption of minimal agreement) it implies affine separate aggregation in the Anscome-Aumann model. In Section 3, the belief-proof Pareto condition is updated for the fully subjective expected utility model. Then, we prove that it is also equivalent to affine separate aggregation. In Section 4, the assumption of minimal agreement is relaxed. Two slightly stronger conditions are introduced and we show that these conditions are also equivalent to separate aggregation. In Section 5, the way these conditions relate to alternative conditions is studied. Especially, we speculate on the meaning of the choice between a 'pure' Savage setting or a 'Savage-Arrow' one.[5] This difference appears to be central while understanding the relation between our results and those of GSS. Finally, the paper closes

[4]It is as if, overall, society (or the social planner) knows individual utilities and priors but, however, is unable to associate each individual with his prior.

[5]A Savage setting endowed with a σ-additive probability measure over events may be called a 'Savage-Arrow' setting.

by pointing out how the belief-proof Pareto condition can be extended to address the aggregation issue within ambiguity models.

2 ANSCOMBE-AUMANN PREFERENCES

Consider the AAEU model. Let X be the set of *outcomes* and Δ be the set of all *simple probability distributions* on X. Let Ω be a set of *states*, which could be either finite or infinite. Let $\mathscr{F}(\Delta)$ be the set of all *simple functions* from Ω to Δ. Society $\mathcal{I}$ is a finite set of I individuals, *i.e.*, $\mathcal{I} = \{1, \ldots, I\}$. Any individual $i \in \mathcal{I}$ has a *preference relation* $\succsim_i \subset \mathscr{F}(\Delta) \times \mathscr{F}(\Delta)$ whereas society's (or social) preferences are denoted by $\succsim_0 \subset \mathscr{F}(\Delta) \times \mathscr{F}(\Delta)$. For all $i \in \{0, 1, \ldots, I\}$, the relations $\sim_i$ and $\succ_i$ are defined as the symmetric and asymmetric part of $\succsim_i$ as usual. It is assumed, for any $i \in \mathcal{I}$, that $\succsim_i$ admits an *expected utility representation*. A function $\mathbf{E}_i : \mathscr{F}(\Delta) \to \mathbb{R}$ is an expected utility function if there exists a continuous affine utility function $u_i : X \to \mathbb{R}$ and a finitely additive probability measure $\pi_i : \Omega \to [0, 1]$ such that, for all $f \in \mathscr{F}(\Delta)$: $\mathbf{E}_i(f) = \int_\Omega u_i(f) \mathrm{d}\pi_i$.[6]

Moreover, a *minimal agreement condition* is assumed (it will be relaxed in Section 4).

MINIMAL AGREEMENT ON THE OUTCOMES (MAO)
For all $0 \leq i \leq I$, there are $z, z' \in X$ such that $z \succ_i z'$.

The Anscombe-Aumann framework is first considered for technical convenience and ease of interpretation.

Exploring the conditions under which social beliefs and utility come from the affine aggregation of individual beliefs and utilities amounts to asking whether there exist nonnegative numbers $\{\alpha_i\}_{i=1}^I$ summing up to one such that $\pi_0 = \sum_{i=1}^I \alpha_i \pi_i$ and nonnegative numbers $\{\beta_i\}_{i=1}^I$ summing up to one such that $u_0 = \sum_{i=1}^I \beta_i u_i$.

In our setting, two cases arise:
(i) individual beliefs are observable and perfectly known by society (or, equivalently, by the social planner): that is the case of *complete information*,
(ii) individual beliefs are no longer observable by society, that is the case *of incomplete information*.

The two next subsections (and that of Section 3, below) are devoted to these two polar situations.

[6]Actually, $\mathbf{E}_i(f) = \int_\Omega \left(\sum_{x \in X} u_i(x) f(\omega)(x) \right) \mathrm{d}\pi_i(\omega)$. By a slightly abusive notation, we also define $u_i : \Delta \to \mathbb{R}$ by $u_i(P) = \sum_{x \in X} u_i(x) P(x)$. Using this definition, $\mathbf{E}_i$ can be written exactly as above.

2.1 Complete information on individual beliefs

Assume that society (or a social planner) knows individual beliefs $\{\pi_i\}_{i=1}^I$ and values $\{u_i\}_{i=1}^I$. In the presence of heterogeneous beliefs, it is important for society to form an acceptable probability measure in order to evaluate every possible act. However, we assume that society does not possess any extra information. Therefore, social beliefs have only to rely on the individual probability estimations. Without taking a stand on which individual beliefs are correct, society could not rule out any individual prior by proving it wrong. Instead, it is natural for society to consider each individual prior being a *reasonable* probability estimation. The insight of the following Pareto condition is that the unanimity condition applies if this unanimity is invariant with respect to every reasonable probability measure—*i.e.*, any individual prior. In that sense, as already said above, it recalls strategy-proofness in game theory since this form of unanimity is immune from any belief manipulation. More specifically, we propose that if each individual prefers an act to another one under every reasonable beliefs, then so does the society. Denote by $\mathbf{E}_{ij} = \int u_i(f)\mathrm{d}\pi_j$, the i's *virtual* expected utility of f with respect to j's prior.

BELIEF-PROOF PARETO CONDITION (AA1)
For any $f, g \in \mathscr{F}(\Delta)$, if $\mathbf{E}_{ij}(f) \geq \mathbf{E}_{ij}(g)$, $\forall i, j \in \mathcal{I}$, then $\mathbf{E}_0(f) \geq \mathbf{E}_0(g)$.

If individuals have common beliefs, as in Harsanyi [1975], that is, if $\pi_i = \pi$ for every $i \in \mathcal{I}$, then the belief-proof Pareto condition (BPPC) coincides with the standard Pareto one. Standard Pareto condition states that if every individual prefers one act to another act, then so does the society. However, unanimity is not compelling for society whenever such a unanimity of preferences are led by conflicting beliefs and values. To rule out spurious unanimities, our criterion requires that every individual should not change her choice between two options even if her prior is replaced by any other individual prior. This principle makes social choices depending not only on the direct preferences comparison, but also on the belief-adjusted preferences comparison—which leaves no room for spurious unanimity.

Theorem 1. BPPC(AA1) *is satisfied iff* π_0 *is an affine combination of* $\{\pi_i\}_{i=1}^I$ *and* u_0 *is an affine combination of* $\{u_i\}_{i=1}^I$.

Proof. The necessity part is straightforward. Therefore, we have just to show the sufficiency one. Suppose BPPC(AA1) is satisfied. It is clear that every individual virtual expected utility $\mathbf{E}_{11}, \ldots, \mathbf{E}_{1I}, \ldots, \mathbf{E}_{I1}, \ldots$ or $\mathbf{E}_{II}$ is an affine function on $\mathscr{F}(\Delta)$. Since the set $\mathscr{F}(\Delta)$ is convex, according to the Harsanyi Theorem (for more rigorous result, see, De Meyer and Mongin [1995]), there exist $\gamma_{ij} \geq 0$ with $\sum_{ij \in \mathcal{I}} \gamma_{ij} = 1$ such that (s.t.), for all $f \in \mathscr{F}$,

$$\mathbf{E}_0(f) = \sum_{i,j \in \mathcal{I}} \gamma_{ij} \mathbf{E}_{ij}(f).$$

Therefore, let $\alpha_i = \sum_j \gamma_{ij}$, then for any $x \in X$:

$$u_0(x) = \sum_{ij} \gamma_{ij} u_i(x) = \sum_i \alpha_i u_i(x).$$

Since there are $z, z' \in X$ s.t. $u_i(z) > u_i(z')$, for all i, we can normalize u_i, without loss of generality, s.t. $u_i(z) = 1$ and $u_i(z') = 0$, for all i. Take any event $E \in \mathscr{A}$. Then,

$$\begin{aligned} \pi_0(E) &= \mathbf{E}_0(zEz') \\ &= \sum_{ij} \gamma_{ij}(u_i(z)\pi_j(E) + u_i(z')\pi_j(E^c)) \\ &= \sum_{ij} \gamma_{ij}\pi_j(E). \end{aligned}$$

Let $\beta_j = \sum_i \gamma_{ij}$. It is then straightforward that $\pi_0 = \sum_j \beta_j \pi_j$. □

Theorem 1 illustrates the relationship between BPPC(AA1) and the separate aggregation rule. Note that BPPC(AA1) is more general than the standard Pareto because the unanimity condition is applied to both I individuals and $I(I-1)$ phantom individuals. In addition, note that it is not always possible to decompose γ_{ij} into α_i and β_j consistently with separate aggregation. Therefore, MAO is necessary since, under this condition, probabilities make become equal to restrictions of suitably normalized expected utility functions. To achieve separate aggregation without MAO, the Pareto condition must be further strengthened. The discussion about this last topic is postponed to Section 4.

2.2 Incomplete information on individual beliefs

The potential usefulness of BPPC(AA1) depends on its ability to allow deducing individual priors from individual preferences. This concern with tractability motivates the analysis of a behavioral expression of BPPC(AA1).

Consider that individuals beliefs are not observable by society, *i.e.,* society's information is incomplete. For any $i \in \mathcal{I}$, given a prior π_i over events, the individual lottery over outcomes X that is induced from a constant act $c \in \Delta$ or a general act $f \in \mathscr{F}(\Delta)$ with respect to π_i is a pure lottery, denoted by λ_i^f, and given by:

$$\begin{cases} \lambda_i^c = c = (x_1, p_1; \ldots; x_n, p_n) & \text{for } c = (x_1, p_1; \ldots; x_n, p_n) \in \Delta, \\ \lambda_i^f = \pi_i(E_1) \cdot c_1 + \cdots + \pi_i(E_n) \cdot c_n & \text{for } f = c_1E_1c_2E_2 \cdots c_nE_n \in \mathscr{F}(\Delta). \end{cases}$$

Since individuals are assumed to subjectively represent the uncertainty they face through probabilities, then, for each act f, there exists an *induced certainty equivalent* (ice) *act*, denoted by c_f^i, such that c_f^i and λ_i^f are identical. An ice-act c_f^i is both a lottery over outcomes induced from f with respect to π_i and a certainty equivalent. Hence, it allows to capture differently belief heterogeneity through certainty equivalents.

Formally, if $f = c_1E_1 \cdots c_nE_n \in \mathscr{F}(\Delta)$, we can identify c_f^i as follows:

$$\begin{aligned} (1) \qquad c_1E_1 \cdots c_nE_n &\sim_i (\alpha_1c_1 + (1-\alpha_1)c_2)E_1 \cup E_2 \cdots c_nE_n \\ &\sim_i [\alpha_2(\alpha_1c_1 + (1-\alpha_1)c_2) + (1-\alpha_2)c_3]E_1 \cup E_2 \cup E_3 \cdots c_nE_n. \end{aligned}$$

Inductively, we get

$$\begin{aligned} f &\sim_i \prod_{i=1}^{n-1} \alpha_i c_1 + \sum_{j=2}^{n-1} (1-\alpha_{j-1}) \prod_{i=j}^{n-1} \alpha_i c_j + (1-\alpha_{n-1})c_n \\ &\equiv c_f^i. \end{aligned}$$

Remark that, since it just involves two constant acts, *i.e.,* c_1, c_2, the value α_1 that can satisfy the very first indifference condition (Eq.1) is the value at which the two acts imply identical induced lotteries—*i.e.,* α_1 should satisfy:

$$u_i(c_1)\pi_i(E_1) + u_i(c_2)\pi_i(E_2) = u_i(\alpha_1c_1 + (1-\alpha_1)c_2)\pi_i(E_1 \cup E_2).$$

Since u_i is an affine function, it implies:

$$\alpha_1 = \frac{\pi_i(E_1)}{\pi_i(E_1) + \pi_i(E_2)}.$$

Similarly, for any $n' = 1, \ldots, n-1$, we have:

$$\alpha_{n'} = \frac{\pi_i(E_1) + \cdots + \pi_i(E_{n'})}{\pi_i(E_1) + \cdots + \pi_i(E_{n'+1})}.$$

Therefore, the induced lottery λ_i^f is equivalent to the ice-act c_f^i. Besides, it is clear that c_f^i is uniquely determined by $\succsim_i$. BPPC(AA1) can then be rewritten accordingly:

BELIEF-PROOF PARETO CONDITION (AA2)
For any $f, g \in \mathscr{F}$, if ice-acts $c_f^i \succsim_j c_g^i$ for all $i, j \in \mathcal{I}$, then $f \succsim_0 g$.

BPPC(AA2) means that, if each individual thinks every reasonable ice-act of f to be preferred to every reasonable ice-act of g, then society prefers f to g. However, ice-acts typically vary across individuals. What unites them all in this context is the fact that each ice-act c_f is induced by a common act f. Since individuals may have different beliefs, these ice-acts may be different from one individual to another. Moreover, it is easy to see that BPPC(AA2) is equivalent to BPPC(AA1): for all $f = c_1 E_1 \cdots c_N E_N \in \mathscr{F}(\Delta)$, and for all $i, j \in \mathcal{I}$, $\mathbf{E}_j(c_f^i) = \int u_j(f)\mathrm{d}\pi_i = \mathbf{E}_{ji}(f)$. Hence, Theorem 1 implies the following result:

Theorem 2. BPPC(AA2) *holds iff π_0 is an affine combination of $\{\pi_i\}_{i=1}^I$ and u_0 is an affine combination of $\{u_i\}_{i=1}^I$.*

This result also proves that, for society, complete information of individual beliefs and values is not necessary to aggregate tastes and beliefs separately.

3 SAVAGE PREFERENCES

We consider now the separate aggregation rule in a Savage framework, namely when both individuals and society are SEU.

Let Ω be a set of states ω and $\mathscr{A}$ be an algebra of events E. Let X be a set of outcomes. An act is a simple[7] and $\mathscr{A}$-measurable function $f : \Omega \to X$. Denote by $\mathscr{F}$ the set of all acts and identify $x \in X$ with the constant act $f = x$. For a sequence $E_1, \ldots, E_n$ of events belonging to $\mathscr{A}$, $f_1 E_1 \cdots f_n E_n g$ denotes the act that pays off $f_i(s)$ if $s \in E_i$ and $g(s)$ if $s \notin \bigcup_{i=1}^n E_i$. Society is a finite set of I individuals: $\mathcal{I} = \{1, \ldots, I\}$. Individual $i \in \mathcal{I}$ has preferences $\succsim_i \subset \mathscr{F} \times \mathscr{F}$, whereas social preferences are denoted by $\succsim_0 \subset \mathscr{F} \times \mathscr{F}$. For all $i \in \mathcal{I}$, the relations $\sim_i$ and $\succ_i$ are defined as the symmetric and asymmetric part of $\succsim_i$ as usual. Assume that all preference relations admit a representation by a subjective expected utility $\mathbf{E}_i : \mathscr{F} \to \mathbb{R}$: for any $i \in \mathcal{I}$, there exist a nonconstant and bounded function $u_i : X \to \mathbb{R}$ and a unique, finitely additive and nonatomic[8] probability measure $\pi_i : \mathscr{A} \to [0, 1]$, such that individual i evaluates every act $f \in \mathscr{F}$ as $\mathbf{E}_i(f) = \int_\Omega u_i(f(\omega))\mathrm{d}\pi_i$. Moreover, as in the preceding section, MAO is assumed.

[7]We say a function f is simple if its outcome set $\{f(\Omega) = \{f(\omega) \mid \omega \in \Omega\}$ is finite.

[8]A probability measure π is said *nonatomic* if, for every event E with $\pi(E) > 0$ and any $\alpha \in (0, 1)$, there exists an event $F \subset E$ such that $\pi(F) = \alpha\pi(E)$.

Note that, while searching for an appropriate form of BPPC for SEU preferences that may imply separate aggregation, the technique used in the proof of Theorem 1 would not be valid either, since the set of acts $\mathscr{F}$ is no longer convex.

3.1 Complete information on individual beliefs

Consider the case where society (or a social planner) has access to individual beliefs $\{\pi_i\}_{i=1}^{I}$, *i.e.*, society's information about individual beliefs is complete.

Given an individual subjective prior π_i over events, for each act $f \in \mathscr{F}$, define again the induced lottery λ_i^f as

$$\lambda_i^f = \left(x_1, \pi_i \circ f^{-1}(x_1); \ldots; x_n, \pi_i \circ f^{-1}(x_n)\right),$$

where $\{x_1, \ldots, x_n\}$ is the outcome set associated with the act f. In words, as in the previous section, λ_i^f is the lottery over outcomes induced by an act f with respect to π_i. Let $\Delta(X)$ the set of simple probabilities on X. Then, the following result holds.

Lemma 1. *For any $i \in \mathcal{I}$, $\{\lambda_i^f = \pi_i \circ f^{-1} \mid f \in \mathscr{F}\} = \Delta(X)$.*

(See Theorem 14.3 in Fishburn [1970] for the proof.)

For any $i \in \mathcal{I}$, according to Lemma 1, if $p \in \Delta(X)$, there always exist an act $f \in \mathscr{F}$ and a prior π_i such that $p = \lambda_i^f$. So, for $p, q \in \Delta(X)$, let f, g be such that $p = \lambda_i^f$ and $q = \lambda_i^g$. Then, we can define a preference relation $\succsim_i^*$ over lotteries $\Delta(X)$ by:

$$p \succsim_i^* q \Longleftrightarrow \lambda_i^f \succsim_i^* \lambda_i^g \Longleftrightarrow f \succsim_i g. \tag{2}$$

To motivate BPPC within SEU, consider the following example. Suppose that individuals $i \in \mathcal{I}$ are unanimous in considering that an act f is preferable to another act g_i with respect to every reasonable beliefs in $\{\pi_i\}_{i=1}^{I}$. That is, for every $i \in \mathcal{I}$, $\lambda_j^f \succsim_i^* \lambda_j^{g_i}$, for all $j \in \mathcal{I}$. Suppose now that there exist two acts $\overline{g}$ and $\underline{g}$ in $\mathscr{F}$ such that g_i can be decomposed into a half-half *subjective mixture of $\overline{g}$ and $\underline{g}$ with respect to π_i, i.e.*, $\lambda_i^{g_i} \equiv 0.5\lambda_i^{\overline{g}} + 0.5\lambda_i^{\underline{g}}$. Equivalently, for every i, $\lambda_j^f \succsim_i^* 0.5\lambda_j^{\overline{g}} + 0.5\lambda_j^{\underline{g}}$, for all $j \in \mathcal{I}$. This means that each individual regards the act f as preferable to a half-half hedging between $\overline{g}$ and $\underline{g}$. Therefore, we suggest that this preference can be adopted by society: $\lambda_0^f \succsim_0^* 0.5\lambda_0^{\overline{g}} + 0.5\lambda_0^{\underline{g}}$.

The following axiom generalizes this example: if all individuals prefer a given subjective mixture of acts $(f_1, \ldots, f_N)$ to another subjective mixture of acts $(g_1, \ldots, g_M)$ with respect to every reasonable beliefs, then so does the society.

BELIEF-PROOF PARETO CONDITION (S1)
For every act $f_n, g_m \in \mathscr{F}$, $n = 1, \dots, N$, $m = 1, \dots, M$, and every nonnegative numbers α_n and β_m such that $\sum_n \alpha_n = 1$ and $\sum_m \beta_m = 1$, if for all $i, j \in \mathcal{I}$,

$$\sum_n \alpha_n \lambda_i^{f_n} \succsim_j^* \sum_m \beta_m \lambda_i^{g_m}, \text{ then } \sum_n \alpha_n \lambda_0^{f_n} \succsim_0^* \sum_m \beta_m \lambda_0^{g_m}.$$

Since every lottery $\lambda_i^{f_n}$ belongs to $\Delta(X)$, all mixtures $\sum_n \alpha_n \lambda_i^{f_n}$ also belong to $\Delta(X)$. Consequently, BPPC(S1) means that, if each individual prefers a hedging of acts to another hedging of acts irrespective of the reasonable beliefs he chooses, then society also prefers the former hedge to the latter one. By Lemma 1, we know that there exists an act whose lottery induced from π_i is the same as $\sum_n \alpha_n \lambda_i^{f_n}$. Actually, these acts typically differ across individual beliefs, as well as social beliefs. However, each of these acts is obtained from the same mixture (α_n) of the same acts (f_n). Note that BPPC(S1) implies BPPC(AA1).

Theorem 3. BPPC(S1) *is satisfied iff π_0 is an affine combination of $\{\pi_i\}_{i=1}^I$ and u_0 is an affine combination of $\{u_i\}_{i=1}^I$.*

Proof. For every $i, j \in \mathcal{I}$ and every $f \in \mathscr{F}$, $\mathbf{E}_{ij}(f) = \int u_i(f) \mathrm{d}\pi_j$. Let $\Delta(\mathscr{F})$ be the set of simple probability distributions over acts. Clearly, $\Delta(\mathscr{F})$ is a convex set. Therefore, for any $f_n \in \mathscr{F}$, $n = 1, \dots, N$, and any nonnegative numbers α_n with $\sum_n \alpha_n = 1$, $\sum \alpha_n f_n \in \Delta(\mathscr{F})$. Define now $\widetilde{\mathbf{E}}_{ij}$ as follows: for every $\sum \alpha_n f_n$, $\widetilde{\mathbf{E}}_{ij}(\sum \alpha_n f_n) = \sum_n \alpha_n \mathbf{E}_{ij}(f_n)$. $\widetilde{\mathbf{E}}_0$ can be defined in a similar way. Hence, consider any pair of acts $\sum \alpha_n f_n$ and $\sum \beta_m g_m$ in $\Delta(\mathscr{F})$ and suppose that, for every $i, j \in \mathcal{I}$, $\widetilde{\mathbf{E}}_{ij}(\sum_n \alpha_n f_n) \geq \widetilde{\mathbf{E}}_{ij}(\sum_m \beta_m g_m)$. By definition, it is equivalent to $\sum_n \alpha_n \mathbf{E}_{ij}(f_n) \geq \sum_m \beta_m \mathbf{E}_{ij}(g_m)$, *i.e.*, $\sum_n \alpha_n \int u_i(f_n) \mathrm{d}\pi_j \geq \sum_m \beta_m \int u_i(g_m) \mathrm{d}\pi_j$. By linearity of the expected utility, it is also equivalent to

$$\int u_i(x) \mathrm{d} \sum_n \alpha_n \lambda_j^{f_n} \geq \int u_i(x) \mathrm{d} \sum_m \beta_m \lambda_j^{g_m},$$

which means that the necessity part of the axiom is satisfied. Then, we must have

$$\int u_0(x) \mathrm{d} \sum_n \alpha_n \lambda_j^{f_n} \geq \int u_0(x) \mathrm{d} \sum_m \beta_m \lambda_j^{g_m},$$

that is: $\widetilde{\mathbf{E}}_0(\sum_n \alpha_n f_n) \geq \widetilde{\mathbf{E}}_0(\sum_m \beta_m g_m)$. We have already proved that the unanimity condition is satisfied with respect to (wrt) $(\{\widetilde{\mathbf{E}}_{ij}\}_{i,j \in \mathcal{I}}, \widetilde{\mathbf{E}}_0)$ on the convex set $\Delta(\mathscr{F})$. Since each $\widetilde{\mathbf{E}}_{ij}$ is an affine function, the Harsanyi Theorem applies: there exist nonnegative numbers γ_{ij} for $i, j \in \mathcal{I}$ s.t., for every $\widetilde{f} \in \Delta(\mathscr{F})$, $\widetilde{\mathbf{E}}_0(\widetilde{f}) = \sum_{ij} \gamma_{ij} \widetilde{\mathbf{E}}_{ij}(\widetilde{f})$. In restricting acts f on $\mathscr{F}$, we have: $\mathbf{E}_0(f) = \sum_{ij} \gamma_{ij} \mathbf{E}_{ij}(f)$. Let $\alpha_i = \sum_j \gamma_{ij}$ and $\beta_j = \sum_i \gamma_{ij}$. As shown in Theorem 1, it is therefore straightforward that $u_0 = \sum_i \alpha_i u_i$ and $\pi_0 = \sum_j \beta_j \pi_j$. $\square$

3.2 Incomplete information on individual beliefs

Separate aggregation seems to be related to the assumption according to which individual beliefs are 'sufficiently' reasonable. What happens now if individual beliefs are not observable by society? In AAEU, we already know that complete information on individual beliefs is not necessary to aggregate tastes and beliefs separately (see above, Theorems 1 and 2). In SEU, the last result (Theorem 3) confirms that, under complete information, BPPC is a necessary and sufficient condition for separate aggregation. Then, the strategy, in case of incomplete information, is intuitively based on the same kind of argument: reformulating utilitarianism in a way that allows separate aggregation of SEU preferences by means of an appropriate version of BPPC. This can be done by means of a detour via several concepts (symmetric and exchangeable partitions, γ-division...) which are needed before writing BPPC in a relevant way for SEU preferences and incomplete information on beliefs. This is the goal of the next definitions and preliminary lemmas.

Definition 1. Given an expected utility preference $\succsim$ on $\mathscr{F}$, for any $f, g \in \mathscr{F}$, a partition $\{E_m\}_{m=1}^M$ is a (f,g)-*symmetric M-partition* if for all $x, x', y, z \in X$ and $m, n \in \{1, \dots, M\}$,

$$x(E_m \cap f^{-1}(y) \cap g^{-1}(z))x' \sim x(E_n \cap f^{-1}(y) \cap g^{-1}(z))x'.$$

For any $1 \leq m \leq M$, the event $E_{m,M} := E_1 \cup \cdots \cup E_m$ is said to be a *m/M-division of* (f,g).

Suppose π is a probability measure on $\mathscr{A}$ with respect to $\succsim$. For any two acts f and g, if there exists a (f,g)-symmetric M-partition $\{E_m\}_{m=1}^M$, then, for all $1 \leq m \leq M$:

$$\pi(E_1) = \cdots = \pi(E_M) = \frac{1}{M}, \text{ while}$$
$$\pi(E_m \cap f^{-1}(x)) = \frac{\pi(f^{-1}(x))}{M} \text{ and } \pi(E_m \cap g^{-1}(y)) = \frac{\pi(g^{-1}(y))}{M}.$$

Therefore, the act $fE_{m,M}g$ would induce a probability distribution over outcomes as following:

$$\pi \circ (fE_{m,M}g)^{-1} = \Big(x_1, \pi(f^{-1}(x_1)) \cdot \frac{m}{M}); \cdots ; x_n, \pi(f^{-1}(x_n)) \cdot \frac{m}{M});$$
$$y_1, \pi(g^{-1}(y_1)) \cdot \frac{M-m}{M}); \cdots ; y_k, \pi(g^{-1}(y_k)) \cdot \frac{M-m}{M})\Big),$$

where $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_k)$ are sets of outcomes associated to f and g, respectively. If the function V represents $\succsim$, then

$$V(fE_{m,M}g) = \frac{m}{M} \cdot V(f) + \frac{M-m}{M} \cdot V(g).$$

The following lemma establishes that any finite symmetric partition always exists.

Lemma 2. *For any $f, g \in \mathscr{F}$ and any $M \in \mathbb{N}$, there exists a (f, g)-symmetric M-partition.*

Proof. This follows easily from the fact that $\{\lambda_i^f = \pi_i \circ f^{-1} \mid f \in \mathscr{F}\} = \Delta(X)$. □

For each individual preference $\succsim_i$, if $x_1, \ldots, x_N$ are in X, we can always find a sequence of events $\{E_n^i\}_{n=1}^{N-1}$ such that event E_n^i is a m_n/M_n-division of $(x_1 E_1^i \cdots x_{n-1} E_{n-1}^i x_n, x_{n+1})$, for every $1 \leq n \leq N-1$.

An act profile $(f_1, \ldots, f_I) = (x_1 E_1^i \cdots x_{n-1} E_{n-1}^i x_n)_{i=1}^I$ is called *rationally identical* if, for all $i \in \mathcal{I}$ and all $1 \leq n \leq N-1$, E_n^i is a m_n/M_n-division of $(x_1 E_1^i \cdots x_{n-1} E_{n-1}^i x_n, x_{n+1})$ with respect to $\succsim_i$. A probability $p \in \Delta(X)$ is said to be *rational* if $p(x)$ is a rational number, for all $x \in X$.

Denote by $\Delta^r(X)$, the set of all rational and simple probabilities.

Lemma 3. *An act profile $(f_1, \ldots, f_I)$ is rational iff there is $p \in \Delta^r(X)$ s.t. $\pi_i \circ f_i^{-1} = p$, for all $i \in \mathcal{I}$.*

Proof. $(\Rightarrow)$ Suppose the act profile $(f_1, \cdots, f_I) = (x_1 E_1^i \cdots x_{N-1} E_{N-1}^i x_N)_{i=1}^I$ is rationally identical. Then, by definition, for all $1 \leq i, j \leq I$:

$$\begin{aligned} \pi_i \circ f_i^{-1} &= \Big(x_1, \prod_{n=1}^{N-1} \frac{m_n}{M_n}; x_2, \frac{M_1 - m_1}{M_1} \cdot \prod_{n=2}^{N-1} \frac{m_n}{M_n}; \ldots; x_n, \frac{M_{N-1} - m_{N-1}}{M_{N-1}}\Big) \\ &= \pi_j \circ f_j^{-1}. \end{aligned}$$

This probability is rational since every m_j and M_j are integers.

($\Leftarrow$) Let $p = (x_1, p_1, \ldots, x_N, p_N) \in \Delta^r(X)$. Clearly, $p_1, \ldots, p_N$ are rational numbers. According to Lemma 2, for $1 \leq i \leq I$, we can find a sequence of events $\{E_n^i\}_{n=1}^{N-1}$ s.t., for each n, E_n^i is a

$$\frac{p_1 + \cdots + p_n}{p_1 + \cdots + p_{n+1}}\text{-division of } (x_1 E_1^i \cdots x_{n-1} E_{n-1}^i x_n, x_{n+1}) \text{ wrt } \succsim_i .$$

Through this construction, it is always possible to find an act profile associated with probability p. $\square$

Definition 2. We say that events E and F in $\mathscr{A}$ are *exchangeable* wrt preference $\succsim$ if E and F are disjoint and, for all $x, x' \in X$ and all $f \in \mathscr{F}$, $xEx'Ff \sim x'ExFf$.

A partition $\{E_m\}_{m=1}^M$ is a *M-exchangeable partition* if, for every $1 \leq m, k \leq M$, the events E_m and E_k are exchangeable.

Definition 3. An act profile $(f_1, \ldots, f_I) = (x_1, E_1^i; \ldots; x_N, E_N^i)_{i=1}^I$ is *identical* if, for any $M \in \mathbb{N}$ and any M-exchangeable partition $\{F_m^i\}_{m=1}^M$ wrt $\succsim_i$, there exists a M-exchangeable partition $\{F_m^j\}_{m=1}^M$, for each $j \neq i$, s.t., for every $y \in X$ and every function $\tilde{f}_k$ defined by:

$$\tilde{f}_k(\omega) = \begin{cases} x_n & \text{if } \omega \in F_m^k \subseteq E_n, \text{ for some } m \\ y & \text{otherwise} \end{cases}, \text{ for every } 0 \leq k \leq I,$$

the profile $(\tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_I)$ is rationally identical.

Lemma 4. *An act profile $(f_1, \ldots, f_I)$ is identical iff there is $p \in \Delta(X)$ s.t. $\pi_i \circ f_i^{-1} = p$, for all $0 \leq i \leq I$.*

Proof. ($\Leftarrow$) Let $p \in \Delta(X)$. For all $1 \leq i \leq I$, let $f_i = (x_1, E_1^i; \ldots, x_N, E_N^i) \in \mathscr{F}$ be s.t. $\pi_i \circ f_i^{-1} = p$. That is, for all $1 \leq n \leq N$ and all $1 \leq i, j \leq I$: $\pi_i(E_n^i) = \pi_j(E_n^j)$. Pick $M \in \mathbb{N}$. Suppose $\{F_m^i\}_{m=1}^M$ is a M-exchangeable partition wrt $\succsim_i$. For every $1 \leq n \leq N$, define $\lambda_n = \#\{m : F_m^i \subseteq E_n^i\}$, *i.e.*, the number of events F_m^i that belong to E_n^i. Therefore, for any $y \in X$, the induced probability of $\tilde{f}_i$ is given by

$$\pi_i \circ \tilde{f}_i^{-1} = \Big(x_1, \frac{\lambda_1}{M}; \ldots; x_N, \frac{\lambda_N}{M}; y, \frac{M - \sum_{n=1}^N \lambda_n}{M}\Big).$$

It is clear that $\pi_i \circ \tilde{f}_i^{-1}$ is a rational probability. To complete the proof of necessity, it suffices to show that for every $j \neq i$, we can construct a M-exchangeable partition s.t. the induced probability of $\tilde{f}_j$ equals to $\pi_i \circ \tilde{f}_i^{-1}$.

Now consider individual $j \neq i$. Since for every $1 \leq n \leq N$, $\pi_j(E_n^j) = \pi_i(E_n^i)$, by nonatomicity of π_j, there exist λ_n disjoint and exchangeable events $E_{n1}^j, \ldots, E_{n\lambda_n}^j \subseteq E_n^j$ s.t.

$$\pi_j(E_{n1}^j) = \cdots = \pi_j(E_{n\lambda_n}^j) = \frac{1}{M}.$$

Repeat this process for all $1 \leq n \leq N$, we can finally find M-exchangeable partition $(E_{11}^j, \ldots, E_{1\lambda_1}^j, \ldots, E_{N1}^j, \ldots, E_{N\lambda_n}^j, \Omega \setminus \cup_{n=1}^N E_{n\lambda_n}^j)$ wrt $\succsim_j$. According to the definition of $\tilde{f}_j$, we have $\pi_j \circ \tilde{f}_j^{-1} = \pi_i \circ \tilde{f}_i^{-1}$.

($\Rightarrow$) Suppose an act profile $(f_1, \ldots, f_I) = (x_1, E_1^i; \ldots; x_N, E_N^i)_{i=1}^I$ is identical. We want to show that there exists $p \in \Delta(X)$ s.t. $\pi_i \circ f^{-1} = p$ for all $0 \leq i \leq I$. We consider two cases.

Case 1: for any $1 \leq i \leq I$, there exists a M-exchangeable partition s.t. $f_i = \tilde{f}_i$. Therefore, $(f_1, \ldots, f_I)$ is rationally identical and Lemma 3 says that there exists a rational probability $p \in \Delta(X)$ s.t. $\pi_i \circ f_i^{-1} = p$, for all $1 \leq i \leq I$.

Case 2: for any $1 \leq i \leq I$, there does not exist a M-exchangeable partition s.t. $f_i = \tilde{f}_i$. Fix $1 \leq i \leq I$. For $M \in \mathbb{N}$, consider a specific way to construct the M-exchangeable partition through $f_i = (x_1, E_1^i, \ldots, x_N, E_N^i)$. Let $p = \pi \circ f_i^{-1}$ with $p_n = \pi(E_n^i)$, for all $1 \leq n \leq N$. For $1 \leq n \leq N$, if

$$\frac{1}{M} \leq \pi_i(E_n^i) \leq \frac{m_n}{M}, \text{ for some } 1 \leq m_n \leq M,$$

then, the nonatomicity of π_i allows to construct m_n disjoint exchangeable events $f_{n1}^i, \ldots, f_{nm_n}^i$ in E_n^i. Let $\bar{m} = M - \sum_{n=1}^N m_n$. Then, the induced probability of the act $\tilde{f}_{i,M}$ is given by:

$$\pi \circ \tilde{f}_{i,M} = (x_1, \frac{m_1}{M}; \ldots; x_N, \frac{m_N}{M}; y, \frac{\bar{m}}{M}).$$

According to the definition of $\tilde{f}_{i,M}$, for all $1 \leq n \leq N$,

$$0 \leq p_n - \pi_i \circ \tilde{f}_{i,M}(x_n) < \frac{1}{M}.$$

Let $\epsilon > 0$. For any $M > \left[\frac{N+1}{\epsilon}\right] + 1$, we have, for any $1 \leq n \leq N$:

$$p_n - \pi_i \circ \tilde{f}_{i,M}(x_n) < \frac{\epsilon}{N+1} \text{ and } \pi_i \circ \tilde{f}_{i,M}(y) < \frac{N \cdot \epsilon}{N+1}.$$

Therefore, $\pi_i \circ \tilde{f}_{i,M}$ weakly converges to p for $M \to \infty$.

Finally, suppose there is some $j \neq i$ s.t. $\pi_j \circ f_j^{-1} = q \neq p$. Since the act profile is identical, for any $M \in \mathbb{N}$, we can always find a M-exchangeable partition s.t. the induced probability of $\tilde{f}_{j,M}$

has a probability distribution identical to that of $\tilde{f}_{i,M}$. In other words, the sequences of probability $\{\pi_i \circ \tilde{f}_{i,M}\}_M$ and $\{\pi_j \circ \tilde{f}_{j,M}\}_M$ are the same. Hence, they must converge to the same probability. Thus, for any $j \neq i$, the induced probability of the act f_j must be p. $\square$

Definition 4. For any $\gamma \in (0,1)$, an event E is a γ*-division of* (f,g) if, for any (f,g)-symmetric M-partition and any $m, m' < M$ with $\frac{m'}{M} \leq \gamma \leq \frac{m}{M}$ and for all constant acts $x \succ x'$ and y, z,

$$x(E_{mM} \cap f^{-1}(y) \cap g^{-1}(z))x'^{-1}(y) \cap g^{-1}(z))x' \succsim x(E_{m'M} \cap f^{-1}(y) \cap g^{-1}(z))x'.$$

Lemma 5. *For $\gamma \in (0,1)$ and acts f, g, if event E is a γ-division of f and g, then $\pi(E) = \gamma$ and*

$$\pi \circ (fEg)^{-1} = \gamma \cdot \pi \circ f^{-1} + (1-\gamma) \cdot \pi \circ g^{-1}.$$

Proof. Suppose $\pi(E) \neq \gamma$. Assume $\pi(E) > \gamma$. Then, by Lemma 2, for acts f, g, there exists an event E_{mM} being the m/M-division of f, g s.t. $\gamma < \frac{m}{M} < \pi(E)$. By Definition 4, we must have, for any $y, z \in X$,

$$\pi(E_{mM} \cap f^{-1}(y) \cap g^{-1}(z)) \geq \pi(E \cap f^{-1}(y) \cap g^{-1}(z)).$$

Since $(f^{-1}(y) \cap g^{-1}(z))_{y,z \in X}$ is a partition of Ω, we have: $\pi(E_{mM}) \geq \pi(E)$. This stands in contradiction to $\frac{m}{M} < \pi(E)$. For the case $\pi(E) < \gamma$, a similar process would also lead to a contradiction. Therefore, $\pi(E) = \gamma$.

Now, let $\{E_m\}_{m=1}^{M}$ be a (f,g)-symmetric M-partition. Let $m, m' < M$ be s.t. $\frac{m'}{M} \leq \gamma \leq \frac{m}{M}$. By definition, for any $y, z \in X$,

$$\frac{m}{M} \cdot \pi(f^{-1}(y) \cap g^{-1}(z)) \geq \pi(E \cap f^{-1}(y) \cap g^{-1}(z)) \geq \frac{m'}{M} \cdot f^{-1}(y) \cap g^{-1}(z)),$$

which leads to

$$\frac{m}{M} \geq \frac{\pi(E \cap f^{-1}(y) \cap g^{-1}(z))}{\pi(f^{-1}(y) \cap g^{-1}(z))} \geq \frac{m'}{M}.$$

So, for any $y, z \in X$,

$$\frac{\pi(E \cap f^{-1}(y) \cap g^{-1}(z))}{\pi(f^{-1}(y) \cap g^{-1}(z))} = \gamma.$$

Therefore, for any $x \in X$,

$$\begin{aligned}\pi(fEg)^{-1}(x) &= \gamma \cdot \sum_z \pi(f^{-1}(x) \cap g^{-1}(z)) + (1-\gamma) \cdot \sum_y \pi(f^{-1}(y) \cap g^{-1}(x)) \\ &= \gamma \cdot \pi(f^{-1}(x)) + (1-\gamma) \cdot \pi(g^{-1}(x)).\end{aligned}$$

Finally, the induced probability of fEg is a γ-combination of the induced probabilities of f and g. $\square$

Lemma 6. *For any $\gamma \in (0,1)$ and any $f, g \in \mathscr{F}$, there always exists an event E that is a γ-division of (f,g).*

Proof. Take any $\gamma \in (0,1)$ and acts f, g. For any $y, z \in X$, by nonatomicity of π on $\mathscr{A}$, there always exists a subevent $E_{yz} \subset f^{-1}(y) \cap g^{-1}(z)$ s.t. $\pi(E_{yz}) = \gamma \cdot \pi(f^{-1}(y) \cap g^{-1}(z))$. Let $E = \cup_{y,z \in X} E_{yz}$. It is straightforward that such an event is indeed a γ-division of f, g. $\square$

Clearly, iterated divisions of acts such as E can be considered as a $1/2$-division of f, g and F as a $1/2$-division of fEg, h, which is then tantamount to consider a $(1/4 : 1/4 : 1/2)$ mixture of f, g, h. More generally, for any nonnegative numbers $\{\gamma_n\}_{n=1}^N$ with $\sum_{n=1}^N \gamma_n = 1$ and acts $f_1, \ldots, f_N$, denote by

$$\sum_{n=1}^{N} \gamma_n \otimes f_n$$

a short-hand for the iterated mixture of acts $f_1, \ldots, f_N$, namely a $\gamma_1, \ldots, \gamma_N$ mixture of $f_1, \ldots, f_N$. Then, for every π_i, we have:

$$\pi_i \circ (\sum_{n=1}^{N} \gamma_n \otimes f_n)^{-1} = \sum_{n=1}^{N} \gamma_n \cdot \pi_i \circ (f_n^{-1}). \tag{3}$$

For any act f_n and any $i \in \mathcal{I}$, denote by $(f_{n,1}, \ldots f_{n,i}, \ldots, f_I)$ the act profile where $f_{n,i} = f_n$. Now, BPPC(S2) can be rewritten accordingly:

BELIEF-PROOF PARETO CONDITION (S2)

For every act $f_n, g_m \in \mathscr{F}$, $n = 1, \ldots, N$, $m = 1, \ldots, M$, and every nonnegative numbers α_n and β_m such that $\sum_n \alpha_n = 1$ and $\sum_m \beta_m = 1$, if for all $i, j \in \mathcal{I}$,

$$\sum_n \alpha_n \otimes f_{n,j} \succsim_i \sum_m \beta_m \otimes g_{m,j}, \text{ then } \sum_n \alpha_n \otimes f_n \succsim_0 \sum_m \beta_m \otimes g_{m,j}.$$

Thanks to the above discussion, the following result is immediate.

Theorem 4. BPPC(S2) *holds iff* π_0 *is an affine aggregation of* $\{\pi_i\}_{i=1}^I$ *and* u_0 *is an affine aggregation of* $\{u_i\}_{i=1}^I$.

This last theorem shows that even in a pure Savagian context, it is not necessary for society to acknowledge individual beliefs or utilities to implement affine aggregation.

4 DISAGREEMENT

In Section 2, a standard notion of minimal agreement (MAO) among individuals was assumed for technical reasons. Here, we want to release it since separate aggregation should sometimes be an appropriate rule for social contexts without minimal agreement. For that purpose, suppose again that individuals are AAEU and society has a complete information on their beliefs. If individual beliefs are known, it is possible for society to transform each act into a lottery. Therefore, individual preferences over acts can be automatically translated into individual preferences over lotteries. Whenever preferences over lotteries are provided, the Pareto condition over lotteries should be applied: that is, unanimous preference comparison for lotteries are compelling. Recall the definition of $\succsim_i^*$ in (2). A first Pareto condition can then be formally stated.

LOTTERY PARETO CONDITION (LPC)
If for all $i \in \mathcal{I}$, $p \succsim_i^* q$, then $p \succsim_0^* q$.

Since LPC coincides with the original Harsanyi's Pareto condition, the utilitarian result follows clearly.

Proposition 1. LPC *is satisfied iff* u_0 *is an affine combination of* $\{u_i\}_{i=1}^I$.

Samuelson [1963] (see also, Arrow [1994]) appears to consider that a failure of individualism questions its normative demand. Consequently, he suggests that it is not necessary for society to fully respect individual beliefs. In this case, LPC could be a plausible condition imposed upon such a society. Hence, society would not only be utilitarian with respect to individual tastes but also free to choose the beliefs it thinks reasonable. Further, if society believes in methodological individualism and respects individual beliefs, it is then rational to regard any individual beliefs as reasonable. Finally, whenever two acts are compared, if this comparison is invariant regardless of any reasonable beliefs, then it is compelling for society to adopt this preference.

SOCIETAL PARETO CONDITION (SPC)
For all $f, g \in \mathscr{F}(\Delta)$, if $\lambda_i^f \succsim_0^* \lambda_i^g$, for all $i \in \mathcal{I}$, then $f \succsim_0 g$.

SPC is actually equivalent to affine aggregation of individual beliefs.

Proposition 2. SPC *is satisfied iff* π_0 *is an affine combination of* $\{\pi_i\}_{i=1}^I$.

Proof. Denote by $\mathbf{E}_{0i}(f) = \int u_0(f)\mathrm{d}\pi_i$, the social *virtual* expected utility of f wrt individual i's prior. SPC can be rewritten: for all $f, g \in \mathscr{F}(\Delta)$, if $\mathbf{E}_{0i}(f) \geq \mathbf{E}_{0i}(g)$, for all $i \in \mathcal{I}$, then $\mathbf{E}_0(f) \geq \mathbf{E}_0(g)$. Thanks to a property of SEU, any $\mathbf{E}_{0i}$ of $\{\mathbf{E}_{0i}\}_{i=1}^I$ is an affine function on the convex set $\mathscr{F}(\Delta)$. Then, according to De Meyer and Mongin [1995], it implies that there exist positive numbers $\{\beta_i\}_{i=1}^I$ with $\sum_i \beta_i = 1$ s.t. $\mathbf{E}_0(f) = \sum_i \beta_i \mathbf{E}_{0i}(f)$. Hence, $\pi_0 = \sum_i \beta_i \pi_i$. $\square$

Actually, these two conditions imply BPPC(AA1), but the opposite is not true. In other words, getting separate aggregation either requires MAO or has recourse to the two above conditions.

Proposition 3. LPC *and* SPC *imply* BPPC(AA1).

Proof. Let $f, g \in \mathscr{F}(\Delta)$. Suppose that, for all $i, j \in \mathcal{I}$, $\mathbf{E}_{ij}(f) \geq \mathbf{E}_{ij}(g)$. Then, for all positive numbers α_i with $\sum_i \alpha_i = 1$, we have for all j, $\sum_i \alpha_i \mathbf{E}_{ij}(f) \geq \sum_i \alpha_i \mathbf{E}_{ij}(g)$. Equivalently,

$$\int \left(\sum_i \alpha_i u_i(f) \right) \mathrm{d}\pi_j \geq \int \left(\sum_i \alpha_i u_i(g) \right) \mathrm{d}\pi_j.$$

According to LPC, this implies $\mathbf{E}_{0j}(f) \geq \mathbf{E}_{0j}(g)$, for all j. Hence, $\mathbf{E}_0(f) \geq \mathbf{E}_0(g)$ follows directly from SPC. $\square$

Following the same technical strategy as in Sections 2 and 3, one can update the different versions for LPC and SPC with various states of information or frameworks.

5 Discussion

5.1 Alternative principles

Under Bayesian environment, where both individuals and society are SEU, many alternative principles to guide society in its choices are found in literature. However, not every principle shares the same methodological approach. To the best of our knowledge, the first alternative principle is suggested by Mongin [1997] which embodies a somewhat different requirement for genuine unanimity preferences: unanimous preference comparisons are considered compelling for society if and only if individuals agree on both probabilities and utility rankings underlying these preferences. Nevertheless, such a homogeneous approach seems to confuse ordinal and cardinal preferences. Actually, ordinal rankings do not convey any quantifiable information about the difference between two evaluations. Therefore, unanimity preferences fail to be compelling for society, insofar as homogeneous ordinal rankings do not constitute a persuasive basis for social reasoning. Alternatively, Billot and Vergopoulos [2016] think that society should not always share the state space with individuals. Consequently, they propose to allow the social space to be defined as the Cartesian product of individual ones. In such a setting, a somewhat extended Pareto principle would lead society to be utilitarian. Another approach is provided by Mongin and Pivato [2016]. They distinguish objective and subjective uncertainty and suggest possible solutions for situations where Pareto conditions can be translated throught weighted additive utility representations.

Finally, the most well-known alternative principle, perhaps the most relevant one for comparison with ours, is given by GSS. Let us begin with a formal restatement of their *restricted Pareto principle*: define an act f as a GSS-*lottery* if $\pi_i \circ f^{-1}(x) = \pi_j \circ f^{-1}(x)$, for all $i, j \in \mathcal{I}$ and all $x \in X$.

RESTRICTED PARETO CONDITION (RPC)
For any pair of GSS-lotteries f and g, if $f \succsim_i g$ for all $i \in \mathcal{I}$, then $f \succsim_0 g$.

First, GSS define GSS-lotteries and then propose to apply the Pareto condition to GSS-lotteries only. One consequence of this approach is that their principle holds only for preference profiles that admit identical probability estimates over at least some events—thus excluding profiles where no such events prevail globally. At a practical level, it appears as a limitation. It is apparent that if one GSS-lottery is unanimously preferred to another one, then, according to BPPC, society should also adopt such a preference. Clearly, BPPC is stronger than RPC. The economic meaning of the difference between these two conditions can be expressed like this: BPPC does not require the existence of events that are equally estimated by all individuals. Moreover, in our view, RPC is a too demanding condition for a reliable interpretation of the Pareto principle: for instance, there may not be GSS-lotteries required by RPC, even when an act unanimously *first order stochastically dominates* another one. In contrast, under BPPC, in case of unanimity preferences respectful of first order stochastic dominance, society would evidently adopt these preferences. Indeed, this is not to dispute the usefulness of RPC. Gilboa et al. [2004] show that RPC is a necessary and sufficient condition for separate aggregation in a 'Savage-Arrow' setting, *i.e.*, a special Savage setting with σ-additive probability measure over events. The extension of the Savage domain to the 'Savage-Arrow' domain is particularly welcome in this context, because σ-additivity guarantees the existence of identical events for all possible probabilities. However, both De Finetti [1974] and Savage [1954] point out that σ-additivity is not a natural requirement for behavior.[9] In fact, the RPC is not equivalent to affine aggregation of individual beliefs if the individual beliefs are not σ-

[9] See Seidenfeld [2001] for a recent discussion of σ-additivity.

additive. At last, a closing remark about the justification of RPC. Actually, GSS implicitly assume that society has a complete information on individual priors and, therefore, could identify GSS-lotteries before applying RPC. This assumption does not seem to be very practical and, moreover, does not cover the case of incomplete information on individual beliefs. Hence, GSS stand in need of a preference-based RPC.

5.2 State-dependent expected utility

A trend in Bayesian theory emphasizes that *state-independence* is not a proper assumption for expected utility theory. Mongin [1998] and Chambers and Hayashi [2006] show that possibility result for aggregation can be achieved if society is a state-dependent expected utility maximizer while Karni [2007] provides a set of axioms characterizing the state-dependent expected utility representation. However, social beliefs are not unique whenever society is of that type. Therefore, the normative stand for belief aggregation is not clear. Also, the results for state-dependent expected utility are developed within Anscombe-Aumann framework only and it is not straight to see what can be achieved in Savage framework.

6 CONCLUDING REMARKS

The choice of a pertinent principle for aggregation of individual preferences has long been the core debate among social choice theorists. This paper propose a new principle which characterizes separate aggregation within AAEU and SEU, under complete or incomplete information on individual beliefs. However, we consider exclusively SEU preferences. Yet, to be satisfactory in itself, an aggregation principle should not be tied to a particular model—not even SEU. Actually, BPPC can be naturally applied to all models in which beliefs and tastes are represented separately. An an illustration, consider application of our principle to the model of *maxmin expected utility* (MEU) (Gilboa and Schmeidler [1989]) and to the model of *Choquet expected utility* (CEU) (Schmeidler [1989]). Firstly, focus on preferences that conform to CEU. This is a more general model than SEU. Therefore, it is appropriate to illustrate the malleability of BPPC and to demonstrate its fundamental relationship with separate aggregation. The same process can be reproduced for MEU and, therefore, we confine ourselves to CEU. Hence, for $0 \leq i \leq I$, preferences $\succsim_i$ are represented by $W_i(f) = \oint_\Omega u_i(f) \mathrm{d}\nu_i$, where $\nu_i : \mathscr{A} \to [0,1]$ is a Choquet capacity and $u_i : X \to \mathbb{R}$, a utility function. Recall now that BPPC requires unanimity preferences to endure interchange beliefs. Based on this, we can write this principle with respect to CEU. Denote by $W_{ij}(f) = \oint u_i(f)\mathrm{d}\nu_j$, the individual i's *virtual* CEU of f with respect to j 's capacity.

BELIEF-PROOF PARETO CONDITION (CEU)

For every act $f_n, g_m \in \mathscr{F}$, $n = 1, \ldots, N$, $m = 1, \ldots, M$, and every nonnegative numbers α_n and β_m such that $\sum_n \alpha_n = 1$ and $\sum_m \beta_m = 1$, if for all $i, j \in \mathcal{I}$,

$$\sum_n \alpha_n W_{ij}(f_n) \geq \sum_m \beta_m W_{ij}(g_m), \text{ then } \sum_n \alpha_n W_0(f_n) \geq \sum_m \beta_m W_0(g_m).$$

The formal logic being similar to that of our previous theorems, we can state the result without proof: BPPC(CEU) *is equivalent to affine separation rules*.

REFERENCES

Shiri Alon and Gabi Gayer. Utilitarian Preferences With Multiple Priors. *Econometrica*, 84(3): 1181–1201, 2016.

Kenneth Arrow. Methodological Individualism and Social Knowledge. *The American Economic Review*, 84(2):1–9, 1994.

Antoine Billot and Vassili Vergopoulos. Aggregation of Paretian preferences for independent individual uncertainties. *Social Choice and Welfare*, 47(4):973–984, 2016.

Markus Brunnermeier, Alp Simsek, and Wei Xiong. A Welfare Criterion For Models With Distorted Beliefs. *The Quarterly Journal of Economics*, 129(4), 2014.

Christopher Chambers and Takashi Hayashi. Preference Aggregation With Incomplete Information. *Econometrica*, 82(2):589–599, 2014.

Christopher P. Chambers and Takashi Hayashi. Preference aggregation under uncertainty: Savage vs. Pareto. *Games and Economic Behavior*, 54(2):430 – 440, 2006.

Hervé Crès, Itzhak Gilboa, and Nicolas Vieille. Aggregation of multiple prior opinions. *Journal of Economic Theory*, 146(6):2563–2582, 2011.

Eric Danan, Thibault Gajdos, Brian Hill, and Jean-Marc Tallon. Robust Social Decisions. *American Economic Review*, 106(9):2407–25, 2016.

Bruno De Finetti. *Theory of probability*, volume 1. Wiley New York, 1974.

Bernard De Meyer and Philippe Mongin. A note on affine aggregation. *Economics Letters*, 47(2): 177–183, 1995.

Peter Fishburn. *Utility Theory for Decision Making*. John Wiley and Sons, 1970.

Gabrielle Gayer, Itzhak Gilboa, Larry Samuelson, and David Schmeidler. Pareto Efficiency with Different Beliefs. *The Journal of Legal Studies*, 43(S2):S151–S171, 2014.

Itzhak Gilboa and David Schmeidler. Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics*, 18(2):141 – 153, 1989.

Itzhak Gilboa, Dov Samet, and David Schmeidler. Utilitarian aggregation of beliefs and tastes. *Journal of Political Economy*, 112(4):932–938, 2004.

Itzhak Gilboa, Larry Samuelson, and David Schmeidler. No-Betting-Pareto Dominance. *Econometrica*, 82(4):1405–1442, 2014.

John Harsanyi. Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. *The Journal of Political Economy*, 63(4):309–321, 1955.

John Harsanyi. Nonlinear Social Welfare Functions: Do Welfare Economists Have a Special Exemption from Bayesian Rationality? *Theory and Decision*, 6:311–332, 1975.

Friedrich A. von. Hayek. *Individualism and Social Order*. University of Chicago Press, 1948.

Aanund Hylland and Richard Zeckhauser. The Impossibility of Bayesian Group Decision Making with Separate Aggregation of Beliefs and Values. *Econometrica*, 47(6):1321–1336, 1979.

Edi Karni. Foundations of Bayesian theory. *Journal of Economic Theory*, 132(1):167 – 188, 2007.

Philippe Mongin. Consistent bayesian aggregation. *Journal of Economic Theory*, 66(2):313–351, 1995.

Philippe Mongin. Spurious unanimity and the Pareto principle. *Economics and Philosophy*, 1: 1–22, 11 1997.

Philippe Mongin. The paradox of the Bayesian experts and state-dependent utility theory. *Journal of Mathematical Economics*, 29(3):331 – 361, 1998.

Philippe Mongin. Spurious unanimity and the Pareto principle. *Economics and Philosophy*, FirstView:1–22, 11 2015.

Philippe Mongin and Marcus Pivato. Ranking multidimensional alternatives and uncertain prospects. *Journal of Economic Theory*, 157:146 – 171, 2015.

Philippe Mongin and Marcus Pivato. Social preference under twofold uncertainty. Working paper, 2016.

Klaus Nehring. The veil of public ignorance. *Journal of Economic Theory*, 119(2):247 – 270, 2004.

Xiangyu Qu. Separate aggregation of beliefs and values under ambiguity. *Economic Theory*, 63 (2):503–519, 2017.

Paul Samuelson. Modern economic realities and individualism. *Texas Quarterly*, 6:128–39, 1963. reprinted in J. Stiglitz, ed., The collected scientific papers of Paul Samuelson, Vol.2. Cambridge, MA: MIT Press, 1966.

Leonard Savage. *The foundations of statistics*. (Second Edition in 1972), New York: Dover, 1954.

David Schmeidler. Subjective Probability and Expected Utility without Additivity. *Econometrica*, 57(3):571–587, 1989.

Teddy Seidenfeld. Remarks on the theory of conditional probability: Some issues of finite versus countable additivity. In *Probability Theory: Philosophy, Recent History and Relations to Science*. Synthese Library, Kluwer, 2001.