Perfect Altruism Breeds Time Consistency

Antoine Billot* and Xiangyu Qu†

Abstract

In this paper, we focus on the process of making economic policies socially acceptable to individuals heterogeneous in terms of discount factors and instantaneous utilities. Through an adapted form of unanimity, we show that *perfect altruism* is the key condition that characterizes a time-consistent society in the presence of heterogeneous individuals. Different levels of altruism intensity are then introduced. We show that they provide different forms of social lifetime utility, from the standard discounted exponential function to quasihyperbolic and k-hyperbolic functions. We also show that by choosing the number of periods involving altruism across unanimity, we can regulate the degree of social present bias. Finally, from these results, new ideas emerge for modifying economic policy recommendations.

1 Introduction

The determinants of general economic decisions, from fiscal policy (Barro [1974]) to environmental policy (Nordhaus [2007]), are often linked in practice to the social discount factor and the social instantaneous utility. The discount factor as well as the instantaneous utility are therefore crucial parameters in the economic evaluation of tax policy, climate policy or economic policy options. The standard approach is described as *paternalistic* and is based, on the one hand, on an exponential discounted utility over time and, on the other hand, on a 'near-one' Ramsey discount factor. However, this method largely neglects individual preferences and, to some extent, is contrary to the spirit of democracy. In particular, in the presence of social heterogeneity, both Zuber [2011] and Jackson and Yariv [2014] show that any nonpaternalistic society violates the Pareto principle of

*Lemma, Université Paris-Panthéon-Assas: billot@u-paris2.fr. A. Billot conducts this research as part of the project Labex MMEDII (ANR 11-LABX-0033-01) for support.

†CNRS, Centre d'Economie de la Sorbonne: xiangyuqu@gmail.com

unanimity, which, according to Buchanan and Tullock [1962], is ethically superior to all other alternative principles. Without the support of any reasonable principle, the legitimacy of paternalism is then widely questioned (see Marglin [1963] and Feldstein [1964]).

In this paper, we first argue that the Pareto unanimity principle is weakly binding when individual preferences are heterogeneous, and we suggest new principles accordingly. We show that under these new conditions, social preferences are compatible with different types of aggregation of individual preferences such that the social discount factor and instantaneous utility take different forms, all derived from individual discount factors and instantaneous utilities, respectively. In contrast to the classic approach, we assert here a nonpaternalistic approach that is, therefore, immune to the question of legitimacy. We seek to provide the foundations for various nonpaternalistic lifetime utilities. Subsequently, our contribution to the field of time preferences develops in several ways. We jointly characterize the social discount factor and instantaneous utility in two contexts: (1) a general time-separable utility (TSU) framework wherein individuals have TSUs and (2) an exponential time-discounted utility (EDU) framework wherein individuals are endowed with EDUs. We identify conditions that allow social entities to be quantified through parameters by aggregating individual entities in a nondictatorial manner, namely, when the utility of each individual influences the formation of social utility.[1]

The EDU model of Ramsey [1928] and Samuelson [1937] has long been recognized as the canonical model of the preferences of a representative agent designed to represent society, although Marglin [1963] and Feldstein [1964] have pointed out the difficulty of deriving a social lifetime EDU by aggregating a society of heterogeneous individuals. In any case, this form is still widely used to evaluate various policies because of its simplicity and, above all, the homogeneity of preferences that it implies. However, the issue of social aggregation has recently introduced new challenges at the academic frontier between theoretical considerations and policy debates. One such challenge is to overcome the unrealistic nature of the homogeneous discounting assumption, which has a profound effect in the context of climate change debates. Optimal climate policy is related to the social value of the future and thus critically depends on the discount factor (Nordhaus [2007]). Investigations by Frederick, Loewenstein, and O'Donoghue [2002] and more recently by Cohen, Ericson, Laibson, and White [2020] illustrate, through different studies and estimates, that individual discount rates differ considerably. As shown by Weitzman [2001] and Drupp et al. [2018], there is no possible convergence to a single impatience rate, even among experts. On the

[1] Weitzman [2001] and Drupp, Freeman, Groom, and Nesje [2018] consider individuals who express support for a near-one discount factor. Thus, a society can place high weight on higher discount factors and maintain intergenerational ethical concerns for long-term projects. In addition, this weighted average method is flexible enough to accommodate the demand for light discounting for short-term projects.

other hand, there is a long-standing traditional belief in economics that individuals differ in their tastes and, therefore, are fundamentally heterogeneous in their instantaneous utility. Heterogeneity is thus not a new challenge for economists, even though it is known to cause both conceptual and theoretical difficulty to be addressed by a consistent aggregation process.

A certain theoretical tradition (Harsanyi [1953]) recommends basing the passage from individual entities to social entities on a rule of aggregation and imposing on this rule the satisfaction of the Pareto principle. As noted by Zuber [2011] and Jackson and Yariv [2014], this literature is specifically devoted to the aggregation of preferences. For Harsanyi [1953], the Pareto principle serves to 'manage' the heterogeneity of tastes but nevertheless fails to produce positive aggregation results when this heterogeneity no longer concerns only tastes but also extends to beliefs. More generally, no possible aggregation can truly occur when individuals are too heterogeneous. In this respect, in terms of time preferences, the standard Pareto condition (PC) alone cannot claim to neutralize the effect of the heterogeneity of individual discount factors when individual instantaneous utilities are already assumed to be heterogeneous and thus cannot provide an axiomatic justification for social time preferences.

Intuitively, it is not absurd to consider that the differences between individual instantaneous utilities and between individual discount factors can cancel each other out in a TSU to produce unanimity. We thus propose an alternative PC called the *impartial PC* (iPC), which states that if all individuals rank one consumption stream higher than another, even when individual discount factors are impartially and arbitrarily permuted, society must approve of this ranking. We then show that this new condition is effective: if both society and individuals have TSUs, iPC gives rise to a social discount factor and a social instantaneous utility that are equal to a weighted average of individual discount factors and individual utilities, respectively.

In the case of an EDU model, which is considered the cornerstone of economic policy analysis, thus requiring principles to justify its theoretical feasibility, this possibility is no longer valid. If we consider these principles from the point of view of a society whose preferences are represented by an EDU, we can show that when individual preferences are heterogeneous, social lifetime utility can be dictatorial even under the iPC. Conversely, a society that is *perfectly altruistic*, *i.e.*, a society wherein the current generation is altruistic only towards the next generation (see Phelps and Pollak [1968]) is compatible with a social EDU. More precisely, the iPC must be adapted to the notion of perfect altruism. Under this *perfectly altruistic* iPC, which restricts comparisons to consumption streams that only differ in the same two periods, we show that a social TSU satisfying a stationarity condition gives rise to a *separately aggregated* EDU: the social entities are defined as the weighted averages of the associated individual entities. In particular, the social discount factor lies between the maximum and minimum of the individual discount factors. One of the main lessons of this result is that perfect altruism, which motivates time consistency, simply corresponds to altruism between any two generations and not necessarily between the current and next generations or even between two successive generations, as prescribed by Barro [1974]. Thus, using Ramsey's ethical critique in favor of a near-one discount factor, many studies have proposed that the planner consider a higher discount factor, situated above the private discount factor of the current generation towards the future (Bernheim [1989], Farhi and Werning [2007], Caplin and Leahy [2004], Feng and Ke [2018]). However, this approach is based on the assumption that a dynastic individual and the planner do not discount the future in the same way. Consequently, this introduces conceptual and theoretical difficulty in justifying the existence of the planner as the result of a process of the aggregation of preferences within society. Recently, Nesje [2021] has shown that this criticism is also valid when society and dynasties have the same preferences for the future.

Finally, as noted by Weitzman [2001], there are situations in which societies behave in a time-inconsistent manner. We therefore propose to study the extent to which deviating from perfect altruism would affect social actualization and, hence, the time consistency of social lifetime utilities. Perhaps surprisingly, the fundamental element of this analysis is that if a social TSU respects an adapted iPC corresponding to an occurrence of *imperfect altruism* with a stationary type condition, then the social discounting factor matches that of the quasihyperbolic discounting model (Phelps and Pollak [1968], Laibson [1997]). More interestingly, we find that as the number of periods involving iPC increases, society is more present biased. This suggests that the degree of social present bias can be regulated by controlling the number of periods involving iPC. However, we are not looking for an abstract specification of the optimal number of periods involving unanimity. Rather, in many empirical situations, we can accept that society is nondogmatic, as suggested by Millner [2020]: society must choose the appropriate number of periods involving iPC, compatible with the problem it has to solve. Harstad [2020] points out, for example, that time inconsistency and strategic investments are crucial for technology policies in the presence of externalities. Thus, once an optimal degree of time inconsistency is determined, society can choose the associated number of periods involving iPC that corresponds to that inconsistency.

The remainder of this article proceeds as follows. Section 2 presents the reference model. Section 3 is devoted to the formal statement of the iPC and its motivations. Section 4 presents the results of the separate aggregation for the case where individual utilities are TSUs, while Section 5 considers the case of a society composed of EDU-type individuals. Next, the results related to the characterization of the time consistency and time inconsistency of social preferences are presented. Section 6 reviews the related literature, and Section 7 concludes the paper. All proofs can be found in the Appendix.

2 The Model

We consider finite society $\mathcal{I}$ consisting of n individuals. Each individual i is assumed to live infinitely and to consume during discrete periods $t \in \mathbb{N} = \{1, 2, \ldots\}$. Let $\mathcal{L}$ be a consumption space, formally a set of lotteries over *finite* outcomes X, *i.e.* , $\mathcal{L} = \Delta(X)$. Each consumption in period t is denoted by z_t and belongs to $\mathcal{L}$. A *consumption stream* is denoted by $\mathbf{z} = (z_1, z_2, \ldots) \in \mathcal{L}^\infty$. For any $z \in \mathcal{L}$, the *constant* consumption stream $(z, z, \ldots)$ is denoted by $\bar{\mathbf{z}}$. For any $t \in \mathbb{N}$ and any $\mathbf{x}, \mathbf{z} \in \mathcal{L}^\infty$, $(x_1, \ldots, x_t, z_1, \ldots)$ is denoted by $\mathbf{x}_t\mathbf{z}$.

2.1 Individual Preferences

Individual preferences over alternative consumption streams are represented by lifetime utility function $U_i : \mathcal{L}^\infty \to \mathbb{R}$, which we assume to be of type TSU, that is, a *time-separable utility*.[2] Namely, for each $t \in \mathbb{N}$ and each $i \in \mathcal{I}$, there exists an individual time-weight or *discount function* d_{it} and a nonconstant and continuous *instantaneous utility* denoted by $u_i : \mathcal{L} \to \mathbb{R}$ such that consumption stream $\mathbf{z} = (z_1, z_2, \ldots) \in \mathcal{L}^\infty$ is evaluated as follows:

$$U_i(\mathbf{z}) = \sum_{t=1}^{\infty} d_{it} u_i(z_t), \tag{1}$$

where $0 < d_{it} < 1$ is (strictly) decreasing in t.[3] In a TSU model, d_{it} depends on time but not on consumption. Without loss of generality (wlog), we normalize $d_{i1} = 1$ for all i. The assumption that d_{it} decreases in t reflects that individuals devalue the future utility flows as a function of the time distance between the present and the future. The positivity of discount factors reflects individual interest in future consumption. The triplet (U_i, u_i, d_i) fully characterizes individual i's TSU.

The most important case of a TSU, which we further discuss below, is the *EDU*. When preferences are for TSU and satisfy Koopmans [1960]'s axioms, they can be represented by an EDU.

---

[2]In fact, a TSU, whether individual or social, is implicitly supposed to depend only on relative time and flow variables and not on absolute time; *i.e.*, it is *time invariant* in the sense of Halevy [2015]. History-dependent lifetime utility would then be an example violating time separability. Moreover, a TSU assumes that the discount function is independent of consumption streams, which rules out, for instance, the *costly empathy* model of Noor and Takeoka [2022].

[3]Since d_{it} is strictly decreasing in t, series $\sum_{i=1}^{\infty} d_{it}$ converges. The fact that outcome set X is finite implies that each u_i is bounded. Hence, $U_i(\mathbf{z})$ is finite for all i and $\mathbf{z}$.

Namely, for $i \in \mathcal{I}$, there exists a *constant* discount factor $\delta_i \in (0,1)$ and a nonconstant and continuous instantaneous utility function $u_i : \mathcal{L} \to \mathbb{R}$ such that consumption stream $\mathbf{z} = (z_1, z_2, \ldots) \in \mathcal{L}^\infty$ is evaluated by i as follows:

$$U_i(\mathbf{z}) = \sum_{t=1}^{\infty} \delta_i^{t-1} u_i(z_t). \tag{2}$$

Hence, triplet (U_i, u_i, δ_i) fully characterizes individual i's EDU.

2.2 Social Preferences

Society must meet three requirements. First, it is assumed that its preferences over streams of consumption, *i.e.*, social preferences, are also represented by a TSU. That is, there exists continuous social instantaneous expected utility u and social discount factor d_t, $0 < d_t < 1$, such that social lifetime utility function $U : \mathcal{L}^\infty \to \mathbb{R}$ is defined by the following:

$$U(\mathbf{z}) = \sum_{t=1}^{\infty} d_t u(z_t). \tag{3}$$

Once again, we normalize $d_1 = 1$, and d_t strictly decreases in t.

The TSU representation is the most general model of preferences satisfying time separability. This model is commonly used for both normative (optimal policy prescription) and positive (description and prediction of behavior) applications. The TSU model includes the hyperbolic discounting model where $d_t = (1+\gamma t)^{-\frac{\alpha}{\gamma}}$ and $\alpha > \gamma$ and the quasihyperbolic discounting model where $d_t = \beta\delta^{t-1}$ for $t > 1$ and many others. Triplets (U, u, d) and (U, u, δ) fully characterize a TSU society and an EDU society, respectively.[4]

Then, society should follow an aggregation rule such that the social discount function and the instantaneous utility are aggregated separately.

Definition 1. A TSU society (U, u, d) admits a *separate* aggregation rule if there exist two functions f and g such that $d = f(d_1, \ldots, d_n)$ and $u = g(u_1, \ldots, u_n)$. In particular, a TSU society admits a *linearly* separate aggregation rule if there exist nonnegative $\{\gamma_i\}_{i\in\mathcal{I}}$ and $\{\alpha_i\}_{i\in\mathcal{I}}$ with $\sum_i \gamma_i = \sum_i \alpha_i = 1$ such that $d = \sum_i \gamma_i d_i$ and $u = \sum_i \alpha_i u_i$.

One can imagine a social TSU such that the social discount function depends on the individual discount functions as well as the individual instantaneous utilities, and similarly for the social

[4]For convenience, we use expressions such as 'the society is a TSU', 'individuals are a TSU' or 'the society is an EDU'.

instantaneous utility function. Since a key part of the theory supporting a TSU in decision-making is based on the fact that the discount and instantaneous utility functions are independently defined, separate aggregation is a natural rule for producing social lifetime utility. Note that assuming the society is a TSU rules out some well-known aggregation rules, notably the utilitarian aggregation rule, based on a weighted average of individual lifetime utilities, and the Rawlsian aggregation rule, based on the minimum of individual lifetime utilities. However, the separate aggregation rule is flexible enough to be compatible with a society satisfying many important properties, such as dynamic consistency and decreasing impatience. For instance, if all individual utilities are EDUs, *i.e.*, $d_{it} = \delta_i^{t-1}$, for all i and t, then a discount function aggregation rule such that $d_t = (\sum_i \gamma_i \delta_i)^{t-1}$ transforms a TSU society into an EDU society, which satisfies dynamic consistency.

Finally, we assume the existence of a *Minimum Agreement over Consumption* (MAC): there exist $x^*, x_* \in X$ such that $u_i(x^*) > u_i(x_*)$ for all $i \in \mathcal{I}$. Therefore, we normalize $u_i(x^*) = 1$ and $u_i(x_*) = 0$ for all i.

3 Pareto Dilemma and Impartiality

The standard PC has long been widely accepted and considered an indisputable reference principle for the aggregation of preferences. However, as suggested by Zuber [2011] and Jackson and Yariv [2015], among many others, the PC is also a source of dilemma in a dynamic setting. Indeed, when both individual and social preferences are assumed to be represented by an EDU, the PC implies society to be dictatorial. Therefore, to better motivate the need to resort to an alternative PC, we first demonstrate that even in our framework, where the lifetime utility of both individuals and society is assumed to be a TSU, the PC and nondictatorship are mutually exclusive.

3.1 Pareto Dilemma

When individual and social preferences are supposed to be represented by a TSU, the PC can be written in the following way:

PC. For any $\mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^\infty$, if $U_i(\mathbf{z}) \geq U_i(\hat{\mathbf{z}})$, for all $i \in \mathcal{I}$, then $U(\mathbf{z}) \geq U(\hat{\mathbf{z}})$.

This condition means that if every individual prefers one consumption stream to another, society does as well. Unfortunately, although Buchanan and Tullock [1962] claim that it is ethically superior to all alternative principles, the PC is basically inconsistent with nondictatorship, which is yet commonly regarded as a minimum imperative for democracy.

In our context, society may display a certain degree of heterogeneity through individual instantaneous utilities and time preferences. Society is said to be *regular* if (i) there are $i, j \in \mathcal{I}$ such that $d_{it} \neq d_{jt}$, for some t, and (ii) individual instantaneous utilities are linearly independent.[5] Finally, society is said to be *dictatorial* if there exists $i \in \mathcal{I}$ such that social and individual i's preferences coincide.

Proposition 1. *Assume society to be regular. Then, the* PC *holds if and only if society is dictatorial.*

Proposition 1 basically means that when individuals are heterogeneous, the Pareto principle of unanimity is equivalent to the existence of a dictator. Although our setting is similar to that of Jackson and Yariv [2015], their result regarding the equivalence between the PC and dictatorship is different from ours.[6] Despite some technical details, these authors assume all lifetime utilities to be EDUs. In contrast, Proposition 1 introduces considerably more flexibility in generalizing Jackson and Yariv [2015]'s negative result to the broader class of TSUs.

3.2 Fictitious Individuals and iPC

Consider the following example to question the legitimacy of the PC. Let a household consist of two individuals, Ana, who is characterized by (u_a, d_{at}), and Bob, who is characterized by (u_b, d_{bt}). If this household is not dictatorial, then wlog, there exists a $\lambda \in (0, 1)$ such that:[7]

$$u = \lambda u_a + (1 - \lambda) u_b.$$

The household is considering whether it is appropriate for it to have a child. If it does not have a child, the household's consumption stream is constant, written as $(x, x, \cdots)$. If it does have a child, its first-period consumption y stands for consumption with the *baby* child, and second-period consumption z stands for consumption with the *adult* child. From the third period, consumption is constant and equal to x. Therefore, the consumption stream with a child is $(y, z, x, x, \cdots)$. The household's instantaneous utility and relative discounting function are presented in the following table:[8]

[5] Recall that $u_1, \ldots, u_n$ are linearly independent if $\sum_i \lambda_i u_i = 0$ implies $\lambda_i = 0$ for all $i \in \mathcal{I}$.

[6] See Theorem 2 of Jackson and Yariv [2015].

[7] Since u_i and u are expected utilities, restricted to constant consumption streams, the Harsanyi Aggregation Theorem (Harsanyi [1955]) requires social instantaneous utility u to be a weighted sum of individual instantaneous utilities.

[8] Since instantaneous utility after the second period is always null, values of the discount function after the second period do not affect the calculation.

$\mathcal{L}$	x	y	z
u_a	0	$\frac{0.98}{\lambda}$	$-\frac{1}{\lambda}$
u_b	0	$-\frac{0.95}{1-\lambda}$	$\frac{9}{1-\lambda}$
u	0	0.03	8

t	1	2
d_a	1	0.99
d_b	1	0.1
d	1	d_2

Ana enjoys time with the baby but worries about the future when the child will have become an adult. As a result, she evaluates y as positive and z as negative. However, Bob finds it boring and expensive to care for a baby but enjoys a priori family happiness once the child grows up. As a result, he evaluates y as negative and z as positive. Furthermore, Ana is very patient and has a high discount value for second-period consumption. In contrast, Bob is very impatient and has a low discount value for second-period consumption. By simple calculation, we have the following:

$$\begin{aligned} U_a(y, z, x, x, \cdots) &= \frac{0.98}{\lambda} - \frac{1}{\lambda} \times 0.99 < 0 = U_a(x, x, \cdots), \\ U_b(y, z, x, x, \cdots) &= -\frac{0.95}{1-\lambda} + \frac{9}{1-\lambda} \times 0.1 < 0 = U_b(x, x, \cdots). \end{aligned}$$

It is straightforward to see that both Ana and Bob prefer not to have a child. However, for any positive household discount function d, we have

$$U(y, z, x, x, \cdots) = 0.03 + 8d_2 > 0 = U(x, x, \cdots).$$

Therefore, *regardless of the discount function*, this household should have a child. This contradiction between the decision produced by the PC and the decision produced by the household's lifetime utility with a utilitarian instantaneous utility reveals that unanimity here is *spurious*.[9] This situation explains why unanimity as formalized by PC violates the interest of the household and, therefore, can hardly be adopted by the household as a fair principle. Intuitively, to avoid such spurious unanimity, Ana and Bob could introduce empathetic considerations. Ana/Bob could replace her/his discounting function with that of Bob/Ana and reconsider her/his choices accordingly. If unanimity remains even by exchanging discount functions, it is no longer spurious but rather *impartial* insofar as each individual discount function is treated impartially, and thus, unanimity can then be considered fair.

A convincing PC must be based on a mutual acceptance of differing opinions. This acceptance can be reflected in a *speculative* experimentation. In terms of time preferences, preferences can be considered unanimous only if, when all individuals replace their own discount function with any other discount function, this permutation never involves a reversal of preferences. The absence of

[9]This notion first appears in Mongin [1995] for an uncertain environment.

preference reversal reveals that such speculative unanimity is robust to any individual discounting. Therefore, a set of all individual discount functions could serve as a common basis for indisputable unanimity.

Formally, any individual i is represented by a pair (d_i, u_i). If, in this pair, d_i is replaced by discount function d_j of individual j, $j \neq i$, we are faced with pair (d_j, u_i) corresponding to abstract individual ji. Since this individual ji, endowed with discount function of j and the instantaneous utility of i, does not correspond to any real individual, we can say that this is fundamentally *fictitious*. Note that only (fictitious) individuals ii for any i are also *actual*. Here, fiction can be seen as an introspective experience involving the association of a discount function and an instantaneous utility that are not jointly observable in actual society: there is no real individual corresponding to fictitious individual ji when $j \neq i$. For convenience, let us call the *fictitious society* Cartesian product $\mathcal{I} \times \mathcal{I}$. Suppose further that the preferences of any fictitious individual ij in $\mathcal{I} \times \mathcal{I}$ over consumption streams are represented by lifetime utility U_{ij} such that for all $\mathbf{z} \in \mathcal{L}^\infty$,

$$U_{ij}(\mathbf{z}) = \sum_{t=1}^{\infty} d_{it} u_j(z_t).$$

The form of U_{ij} expresses how individual j evaluates alternative consumption streams if he replaces his own discount factor with that of individual i.

Let us now introduce a modified PC that takes all fictitious individuals into account.

iPC. For any $\mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^\infty$, if $U_{ij}(\mathbf{z}) \geq U_{ij}(\hat{\mathbf{z}})$, for all $ij \in \mathcal{I} \times \mathcal{I}$, and then $U(\mathbf{z}) \geq U(\hat{\mathbf{z}})$.

This modified PC means that for each pair of consumption streams, if all fictitious individuals unanimously prefer one stream to the other, then so does society. Intuitively, the iPC is reminiscent of the *impartial observer* principle of Harsanyi [1953]. A simple way to understand Harsanyi's intuition about impartiality is as follows: to help choose among social alternatives, each individual is supposed to imagine himself as an impartial observer who knows neither what discounting function nor what instantaneous utility will be assigned to him. Therefore, the impartial observer considers not only the actual preferences but also the fictitious preferences associated with $\mathcal{I} \times \mathcal{I}$. When an impartial observer imagines herself as fictitious individual ij, she then adopts the discount function of i and the instantaneous utility of j to form her preferences. Hence, what is called the *acceptance principle* in the Harsanyi framework plays a role equivalent to that of the iPC in our framework.[10] They can therefore be interpreted in a similar way. Moreover, the iPC introduces the

[10]Harsanyi's acceptance principle states that when the impartial observer imagines herself as individual i, she should adopt i's preferences.

idea that everyone can infer information about other individuals by observing their parameters. The heterogeneity of individual discount functions illustrates, in this respect, that information about the discount function of others is also heterogeneous. From this perspective, society should encourage individuals to learn from each other and gradually eliminate the role of actual individuals in favor of fictitious ones. This can be seen, in a way, as a nondogmatic process in the sense of Millner [2020].

When comparing the iPC and PC, one essential difference appears. With the iPC, society constructs unanimous preferences from all possible fictitious preferences, not just actual ones. Each individual is required to re-evaluate each stream through individual discounting functions other than his or her own to ensure that unanimity is fully convincing. Impartial introspection, which considers anyone else's discounting factor as an introspective experience for oneself, can effectively eliminate occurrences of spurious unanimity, the type of unanimity that is induced by a double mismatch of instantaneous utilities and time preferences.

One could certainly call for respect for the principle of *methodological individualism* and argue that society should base its choices on real, not fictional, individual decisions. It should be emphasized that the iPC is one element of a more general theory of society, *i.e.*, an attempt to understand the forces at work that should determine social decisions. Therefore, the iPC can be seen as an '*ought*'-proposition rather than an '*is*'-proposition (in the words of David Hume). Later in the paper, however, we propose an alternative principle that can be considered this time as an '*is*'-proposition.

4 Separate Aggregation of Time Preferences

In this section, we consider situations wherein the social time discount function and social instantaneous utility are separately aggregated. The next theorem clarifies the connection between the iPC and separate aggregation.

Theorem 1. The iPC *holds if and only if society follows a linearly separate aggregation rule.*

Theorem 1 states that if the iPC is satisfied, social functions (utility and discount) necessarily take the form of a convex combination of individual functions. In contrast to impossibility results such as Proposition 1 and that of Jackson and Yariv [2014], the iPC weakens the PC in a way that avoids spurious unanimity. Hence, this naturally gives rise to a theoretical possibility. To see how the iPC works, note that it requires unanimity with respect to the *fictitious society*, which implies that there exists a nonnegative λ_{ij} such that for a consumption stream $\mathbf{z}$,

$$U(\mathbf{z}) = \sum_{ij} \lambda_{ij} U_{ij}(\mathbf{z}).$$

Then, it can easily be shown that for $\alpha_i = \sum_j \lambda_{ij}$ and $\gamma_j = \sum_i \lambda_{ij}$, society follows the linearly separate aggregation rule.

Indeed, one could argue, for example, that social decisions to protect the environment are in practice based only on actual valuations, not on fictitious ones. Moreover, it is plausible to aggregate individual preferences separately through a Pareto-like condition based solely on individual choices. Assume two consumption streams $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$. If $\mathbf{z}$ and $\mathbf{z}'$ are such that (i) they are constant or (ii) $u_i(z) = u_j(z)$, for all $z \in \text{conv}(\{z_t \mid t \in \mathbb{N}\} \cup \{z'_t \mid t \in \mathbb{N}\})$ and all $i, j \in \mathcal{I}$, then $\mathbf{z}$ and $\mathbf{z}'$ are said to be of *common value*. In other words, two streams are of 'common value' if they are constant or if all individuals have the same instantaneous utilities on the possible outcomes. Note that because of the MAC, there are always nonconstant common-value streams.

common-value PC (cvPC). For any pair of common-valued streams $\mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^\infty$, if $U_i(\mathbf{z}) \geq U_i(\hat{\mathbf{z}})$, for all $i \in \mathcal{I}$, then $U(\mathbf{z}) \geq U(\hat{\mathbf{z}})$.

This modified PC states that for each pair of common-value streams, if all individuals prefer the former over the latter, then so does society. This condition strengthens the PC by focusing on pairs of streams whose values are either constant or identical across all consumption streams. Note that the iPC implies the cvPC simply because the preferences of fictitious individuals ij on common-value streams coincide with those of actual individuals i. The following result shows that the cvPC and MAC also imply linearly separate aggregation.

Theorem 2. The cvPC *holds if and only if society follows a linearly separate aggregation rule.*

Theorem 2 shows that linearly separate aggregation is possible if society satisfies the cvPC. Although both the iPC and cvPC can equivalently imply a separate aggregation rule, special emphasis will be placed on the iPC in the remainder of the analysis because it is, in our view, normatively stronger. However, all of the following results based on the iPC can be equivalently obtained with the cvPC.

A separate aggregation result cannot be used to specify any of the properties of the social discount function. Let us now focus on two important classes of discount functions: the *constant-impatience discount function* and the *present-bias discount function*, both of which reflect decreasing impatience. (We use the terms 'decreasing impatience' and 'present-bias' interchangeably from now on, since it is widely accepted that the former is a testable implication of the latter.) The key property of the first class is that constant impatience generates social choices that are always *time consistent*, which is not only normatively plausible but also highly applicable because of its instrumental simplicity. This leads us to formulate the following question: is there a way to characterize a social discounting function with constant impatience? Moreover, at least since the time of Thaler [1981]'s study, the finding that decision-makers become present biased as the delay increases is a canonically descriptive result. Therefore, one might ask what a social criterion for causing such a present bias looks like.[11] Therefore, we attempt to determine whether this criterion, while generating a social discounting function, defines decreasing impatience behavior on the basis of a reasonable definition of present bias. Such a criterion should not only identify discounting functions but also clarify the behavioral principles underlying the collective decision-making process.

Let $(x_t, \bar{\mathbf{x}}_{*-t})$ denote a consumption stream with $z_t = x$ and $z_s = x_*$ for $s \neq t$.

Definition 2. Lifetime utility $U : \mathcal{L} \to \mathbb{R}$ is *present biased* (resp. *constant impatient*) if for any $t > s$, any $k \geq 1$, $U(x_t, \bar{\mathbf{x}}_{*-t}) = U(y_s, \bar{\mathbf{x}}_{*-s})$ implies $U(x_{t+k}, \bar{\mathbf{x}}_{*-(t+k)}) \geq U(y_{s+k}, \bar{\mathbf{x}}_{*-(s+k)})$ (resp. $U(x_{t+k}, \bar{\mathbf{x}}_{*-(t+k)}) = U(y_{s+k}, \bar{\mathbf{x}}_{*-(s+k)})$).

Lifetime utility is present biased if, once proximate consumption x at time s and additional consumption y at time t are indifferent, additional consumption y is preferred when the two consumption streams are shifted further away at time k. Intuitively, if such a time shift does not change preferences, this lifetime utility is said to be constant impatient.

Next, a present-biased U can also be characterized by its discount function. A *discount factor* measured on date t of a TSU with discounting function d_t, *i.e.* , $\delta_d(t)$, is defined as

$$\delta_d(t) = \frac{d_{t+1}}{d_t}.$$

The following lemma expresses that if a lifetime utility is present biased, its discount factor is increasing. Similarly, if a lifetime utility is constant impatient, its discount factor is constant.[12] This result is stated without proof because of its triviality.[13]

Lemma 1. *Let a TSU be characterized by (U, u, d_t). Then, U is present biased if and only if its discount factor δ_d is increasing. Moreover, U is constant impatient if and only if its discount factor δ_d is constant.*

[11] For the relationship between social decreasing impatience and social present bias, see Jackson and Yariv [2015].

[12] In the case of an EDU, $d_{t+1}/d_t = \delta_d^t/\delta_d^{t-1} = \delta_d$.

[13] Recently, Chambers, Echenique, and Miller [2021] developed a similar definition of decreasing impatience and characterized it in several ways. However, the focus of our paper is different from theirs.

Since we exclusively consider two types of lifetime utility functions, a natural question to ask is the following: is a society composed of constant-impatient or present-biased individuals and with lifetime preferences governed by the iPC necessarily present biased? The next proposition provides a positive answer to this question.

Proposition 2. *Assume that social lifetime utility U follows a linearly separable aggregation rule. If each individual is either constant impatient or present biased, then nondictatorial social utility U is necessarily present biased.*

Proposition 2 can be interpreted as a result demonstrating the nature of the relationship between nondictatorship and present bias when a society is composed of constant-impatient or present-biased individuals and follows a linearly separate aggregation rule. This proves that if the domain of individual lifetime utilities is restricted to only constant-impatient or present-biased TSUs, the iPC implies that society is also present biased. Therefore, contrary to Jackson and Yariv [2015], this result establishes that a present-biased society does not necessarily rely on the assumption that it is composed of constant-impatient individuals. Moreover, a present-biased society does not imply that the individuals in it are either constant impatient or present biased. One can easily construct an example with a first individual who is present biased and a second who is increasingly impatient. If one assigns a sufficiently low weight to the latter, society can still be present biased.

5 ALTRUISM AND SOCIAL IMPATIENCE

In this section, all individual preferences are assumed to be EDUs, $(U_i, u_i, \delta_i)_{i\in\mathcal{I}}$, and social preferences are assumed to be represented by a TSU (U, u, d_t).[14]

5.1 Perfect Altruism and Constant Social Discounting

Although separate aggregation is compatible with the iPC, the question of how dynamically consistent a society can be collectively remains. Since "*the simplicity and elegance of this* (EDU) *formulation is irresistible*", as claimed by Frederick et al. [2002], it is valuable to propose a principle that would characterize a society admitting an EDU representation for its preferences. Here,

[14] Substantial empirical evidence supports the view that individuals do not behave as EDU maximizers when making decisions involving time trade-offs. Indeed, as noted by Frederick et al. [2002], other more realistic types of behavior, such as hyperbolic discounting, are typically assumed. However, since our reasoning essentially revolves around common goods, individual preferences may well not be the same for private goods. Moreover, it is not clear why the hyperbolic discounting behavior of private consumption should inform the discounting assumption for the commons.

we show that an appropriately modified iPC would imply a time-consistent society whose instantaneous utility and discount factor are respectively defined as the convex combination of individual utilities and individual discount factors.

Consider first the stationarity axiom, which is required to ensure constant discounting, as shown by Koopmans [1960].

Stationarity. Lifetime utility function U is *stationary* if for all $x \in \mathcal{L}$ and all $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$,

$$U(\mathbf{z}) \geq U(\mathbf{z}') \text{ if and only if } U(x, \mathbf{z}) \geq U(x, \mathbf{z}').$$

Stationarity means that the ranking of two streams remains unchanged when a common consumption is inserted in the first period of both streams. A decision-maker who obeys this axiom should be insensitive to the inserted consumption. This axiom requires that the evaluation of two consumption streams does not change if all dates are shifted by the same time constant. However, the iPC and stationarity are not sufficient to characterize constant discounting for a nondictatorial social lifetime utility.[15] Therefore, an even more weakened PC is required to derive constant social actualization.

Two consumption streams $\mathbf{z}$ and $\mathbf{z}'$ are said to be *diperiodic* if $z_t = z'_t$, for $t > 2$. In other words, a pair of diperiodic consumption streams can only differ in the first two periods.

perfectly altruistic impartial PC (paiPC). For any two diperiodic consumption streams $\mathbf{z}$ and $\mathbf{z}'$, if $U_{ij}(\mathbf{z}) \geq U_{ij}(\mathbf{z}')$, for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(\mathbf{z}) \geq U(\mathbf{z})$.

This new condition, the paiPC, states that while comparing two consumption streams that only differ in their first two consecutive periods, if all actual and fictitious individuals prefer the first stream to the second, then so does society.

Phelps and Pollak [1968] reminds us that the intuition that individual preferences may be linked by a kind of 'generational' commitment already exists in Ramsey [1928], who assumes that the relationship of each generation's preferences for its own consumption to the preferences of the next generation is identical to the relationship of any future generation's preferences to those of the generation after it. This commitment is equivalent to a stationarity assumption: the preferences of the current generation with respect to consumption streams are assumed to be invariant to changes in timing. Phelps and Pollak [1968] suggests calling this phenomenon *perfect altruism*. A few

[15] Note that the iPC is equivalent to the PC when individual instantaneous utilities are identical, and Jackson and Yariv [2015] demonstrate that there is no nondictatorial social lifetime EDU if the PC and stationarity are imposed on a society whose individuals are heterogeneous in their discounting factors.

years later, Barro [1974]'s analysis of debt neutrality would rely on a similar assumption: that individuals are motivated by a particular form of intergenerational altruism (called here *dynastic altruism*) such that individuals altruistically care for their children, who in turn also have altruistic concerns for their own children, and so on.[16]

By equating a period with a generation, the restriction imposed by the paiPC to ensure time consistency is precisely designed to avoid imperfect altruism between generations since imperfect altruism leads to a violation of stationarity. Considering only the first two consumptions, the lifetime utility of each individual i is then defined as the discounted utility of i and his immediate descendant. Therefore, the social lifetime utility of the first two periods is also the discounted utility of the current and next generations. Stationarity further implies that the social lifetime utility would be recursively evaluated as the discounted sum of all future utilities for which the discount factor is constant.

We can now state one of our main results. If social preferences are represented by a TSU satisfying stationarity and respecting the paiPC, then the social lifetime utility is an EDU. Moreover, the social instantaneous utility and social discount factor are equal to the weighted averages of the individual instantaneous utilities and individual discount factors, respectively.

Theorem 3. Stationarity *and* paiPC *hold if and only if the social lifetime utility is an EDU in which* u *is a convex combination of* $\{u_i\}_{i\in\mathcal{I}}$ *and the social discount function is exponential,* i.e., $d_t = \delta^{t-1}$ *for all* $t \in \mathbb{N}$, *with* δ *being a convex combination of* $\{\delta_i\}_{i\in\mathcal{I}}$.

Theorem 3 means that to be consistent over time, social lifetime utility must be an EDU. Therefore, society must respect both stationarity and the paiPC. In fact, in this situation, the social discount factor can only lie between the minimum and maximum of the individual discount factors, and the social instantaneous utility is the weighted sum of the individual instantaneous utilities. Thus, the exact value of the social discount factor and the exact form of the social utility function depend only on the choice of weights. It should be noted that the weights of the discount factors may differ from those affecting utilities. This means that society may trust the judgment of individual i with respect to his perception of time and give high weight (or even a total weight) to his discount factor but may be more concerned about the well-being of individual j and therefore give more weight (or even a total weight) to his instantaneous utility. In other words, society can arbitrate locally between the discount factor and individual well-being and generalize this arbitration to all individuals.

[16]Due to this recursive relationship, all generations of a family (*i.e.*, a *dynasty*) are linked together through a chain of private intergenerational transfers, preventing any attempt by government to redistribute resources among them.

Since the paiPC restricts comparisons to streams that only differ in the first two periods, it is conceivable to strengthen this condition and, thus, to remove stationarity. For example, stream restrictions can be relaxed to streams that differ in any two arbitrary successive periods: *i.e.*, for any $t \in \mathbb{N}$, any $x, y, x', y' \in \mathcal{L}$, and any $\mathbf{z} \in \mathcal{L}^\infty$ if $U_{ij}(x_t, y_{t+1}, \mathbf{z}_{-(t,t+1)}) \geq U_{ij}(x'_t, y'_{t+1}, \mathbf{z}_{-(t,t+1)})$ for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(x_t, y_{t+1}, \mathbf{z}_{-(t,t+1)}) \geq U(x'_t, y'_{t+1}, \mathbf{z}_{-(t,t+1)})$. In view of a recursive evaluation of welfare for every pair of successive generations, a natural question is whether this version of the iPC along with recursive evaluation would imply stationarity of the social lifetime utility U. In other words, in this situation, is stationarity redundant?

The following example proves that stationarity is not useless. Consider a society of 2 individuals $\{1, 2\}$. Suppose that individuals have identical instantaneous utilities but that their discount factors differ, *i.e.*, $\delta_1 \neq \delta_2$. Suppose hence that society has the same instantaneous utility as individuals and adopts the following discount function:

$$d(t) = \frac{1}{t}\delta_1 + (1 - \frac{1}{t})\delta_2.$$

Clearly, this society does not have a constant discounting factor. Therefore, associated social lifetime utility U violates stationarity. However, it is clear that U satisfies the paiPC. In fact, with stationarity, the above alternative PC turns out to be equivalent to the paiPC.

Now, another way of modifying the paiPC is to relax the requirement that the two periods under consideration be successive. In other words, altruism would no longer be limited to the next generation alone but would instead jump to a later generation. It may happen, for example, that individuals do not care about their children but only about their grandchildren. Is this *postponed altruism* also 'perfect' in the sense of time consistency? Surprisingly, as proven below in Proposition 3, the answer is positive. Let us first adapt the iPC to capture the idea of postponed altruism. Let $k \geq 2$. We say two consumption streams $\mathbf{z}, \mathbf{z}'$ are k-*diperiodic* if $z_t = z'_t$ for $t \in \mathbb{N} \setminus \{1, k\}$.

k-perfectly altruistic impartial PC (k-paiPC): Let $k \in \mathbb{N}$. For any k-diperiodic streams $\mathbf{z}, \mathbf{z}'$, if $U_{ij}(\mathbf{z}) \geq U_{ij}(\mathbf{z}')$, for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(\mathbf{z}) \geq U(\mathbf{z}')$.

This condition, k-paiPC, requires that impartial unanimity applies only if the compared streams differ for the current and k-th generations. With stationarity, we can then prove that this also implies a time-consistent society. Furthermore, the social lifetime utility and social discount factors are weighted averages of individual utilities and factors.

Proposition 3. Stationarity *and* k-paiPC *hold if and only if the social lifetime utility is an EDU in which* u *is a convex combination of* $\{u_i\}_{i \in \mathcal{I}}$ *and* $d_t = \delta^{t-1}$ *for all* $t \in \mathbb{N}$*, with* δ *being a convex combination of* $\{\delta_i\}_{i \in \mathcal{I}}$.

Proposition 3 means that if a stationary social lifetime utility evaluates individual welfare such that society is only concerned about the utilities of the current generation and the k-th generation, then the lifetime utility of this society is an EDU. Relative to Theorem 3, where the paiPC is assumed along with stationarity, Proposition 3 leads to the same utilitarian characterization while assuming the k-paiPC and stationarity. Without delving into the formal proof, to be convinced of this, it is sufficient to consider a situation where individual utilities are identical. Therefore, k-paiPC implies that the value of the social discount function at time k is a weighted average of $\{\delta_i^{k-1}\}_{i\in\mathcal{I}}$. Since this average is between $(\max_{i\in\mathcal{I}} \delta_i)^{k-1}$ and $(\min_{i\in\mathcal{I}} \delta_i)^{k-1}$, there should exist $\delta \in [\min_{i\in\mathcal{I}} \delta_i, \max_{i\in\mathcal{I}} \delta_i]$ such that δ^{k-1} corresponds exactly to that weighted average value. Stationarity further implies that this society admits a lifetime utility with a constant discounting factor.

Proposition 3 has surprising implications. To be time-consistent, society needs only to consider the utilities of two distinct generations that are not necessarily successive. This is equivalent to saying that a society affected by a distant generation can be regarded as a society affected by the next generation. In a way, this corrects the prevalence of the belief that a perfectly altruistic society is concerned only with the utility of immediate children and not that of distant descendants.

5.2 Quasihyperbolic Social Discounting

Although time consistency is attractive for economic theory, it is nevertheless not very observable in economic policy. It is, for example, recognized that the principle of political rotation in a democracy leads to a certain amount of time inconsistency, which invalidates the hypothesis of a social EDU (see, *e.g.*, Harstad [2020]). This can be explained either by the fact that society does not have the real power to make the commitment or by the fact that the benefits it could derive from this commitment are much lesser than the associated costs. The quasihyperbolic discounting model developed by Phelps and Pollak [1968] and Laibson [1997] has long served as the standard reference for the economic analysis of time inconsistency. Although the potential implications of this inconsistency have been frequently noted, few studies have addressed the mechanisms by which preference aggregation leads to quasihyperbolic discounting.

Definition 3. Lifetime utility $U : \mathcal{L}^\infty \to \mathbb{R}$ admits a *quasihyperbolic discounting* form if there exists continuous function u on $\mathcal{L}$ and parameters $\beta \in (0, 1]$ and $\delta \in (0, 1)$ such that for $z \in \mathcal{L}^\infty$,

$$U(\mathbf{z}) = u(z_1) + \beta \sum_{t=2}^{\infty} \delta^{t-1} u(z_t). \tag{4}$$

Of particular interest is the question of the principles that society should respect for social lifetime utility to admit a quasihyperbolic discounting form. From such a social lifetime utility being time-inconsistent, we already know that it violates stationarity. Hence, a weaker stationarity-like condition is required:

Quasi-stationarity. Lifetime utility U is *quasistationary* if for all $x, y \in \mathcal{L}$ and all $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$,

$$U(x, \mathbf{z}) \geq U(x, \mathbf{z}') \text{ if and only if } U(x, y, \mathbf{z}) \geq U(x, y, \mathbf{z}').$$

This condition means that the social evaluation of consumption streams with respect to the next period is insensitive to changes that may occur in future periods. Thus, quasi-stationarity admits the possibility that society may assign to current consumption a place of relative importance out of proportion to all future consumption. It is clear that stationarity implies quasi-stationarity (but not vice versa). Therefore, it is intuitive that quasi-stationarity and the paiPC are compatible with quasihyperbolic social actualization. However, in this situation, β and δ in (4) are indeterminate. From the above analysis, quasi-stationarity and the paiPC imply that the product of β and δ is a weighted average of the individual discount factors. Therefore, society can freely choose whether or not β or δ is a weighted average. Moreover, such indeterminacy contradicts the democratic intuition that each individual should have a say in every social issue. Hence, to avoid this indeterminacy, a stronger condition than the paiPC is required. We say that two consumption streams $\mathbf{z}, \mathbf{z}'$ are *triperiodic* if $z_t = z'_t$, for $t > 3$.

quasialtruistic impartial PC (qaiPC). For any pair of two triperiodic consumption streams $\mathbf{z}, \mathbf{z}'$, if $U_{ij}(\mathbf{z}) \geq U_{ij}(\mathbf{z}')$, for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(\mathbf{z}) \geq U(\mathbf{z})$.

This condition states that, assuming that two consumption streams differ only in the first three periods, the fact that all actual and fictitious individuals rank these two streams in the same way implies that society also adopts this ranking. In this situation, society cares directly about the next two generations, which, in the spirit of Phelps and Pollak [1968], reflects imperfect altruism. However, this imperfection is not only compatible with quasihyperbolic social actualization but also resolves the indeterminacy of β and δ.

Theorem 4. Quasi-stationarity *and* the qaiPC *hold if and only if there exist nonnegative* $\{\alpha_i\}_{i\in\mathcal{I}}$ *and* $\{\lambda_i\}_{i\in\mathcal{I}}$ *with* $\sum_i \alpha_i = \sum_i \lambda_i = 1$ *such that social lifetime utility admits a quasihyperbolic discounting form as defined in (4), with:*

$$u = \sum_i \alpha_i u_i \quad \text{and} \quad \delta = \frac{\sum_i \lambda_i \delta_i^2}{\sum_i \lambda_i \delta_i} \quad \text{and} \quad \beta = \frac{(\sum_i \lambda_i \delta_i)^2}{\sum_i \lambda_i \delta_i^2}.$$

Furthermore, $\delta \in [\min_{i\in\mathcal{I}} \delta_i, \max_{i\in\mathcal{I}} \delta_i]$ *and* $\beta \in [\frac{\min_i \delta_i}{\max_i \delta_i}, 1]$.

Theorem 4 shows that if a society respects the qaiPC and quasi-stationarity, then the social lifetime utility has a quasihyperbolic discounting form. The social instantaneous utility is a weighted average of individual utilities, while social discount factor δ is a proportion of the second moment to the first moment of the individual discount factors, and the social present bias is a proportion of the square of the first moment to the second moment. This result proves that a deviation from perfect altruism leads to present bias. There is a clear trade-off between present bias and impatience with future generations. If society is less present biased, it might allow more variety in the weights affecting individuals. However, since β is close to one, society tends to impose the discount factor, that is, $\delta \approx \delta_i$, for each individual i. Furthermore, the extent of β is determined by the degree of heterogeneity in the discount factors. If individuals are more diverse in terms of impatience, society will be more present biased. This highlights the source of the present bias, namely, the heterogeneity of the individual discount factors.

As done in Subsection 5.1 for the paiPC, the qaiPC can also be relaxed, allowing the compared streams to be different in three periods that are not necessarily consecutive. Note that these three arbitrary periods must include the current period to reflect trade-offs between current and future generations. This difference is measured by β.

5.3 Social Delayed Stationarity

Constant and quasihyperbolic social discounting are compatible with paiPC and qaiPC, respectively. This naturally gives rise to the following question: what would social discounting involve if we were to force this extension to a more general level at which altruistic concern is extended to k-th generations? We have already observed that the qaiPC triggers dynamic inconsistency. Thus, it would not be surprising if spreading altruism also led to such inconsistency. A deeper issue is that this *imperfect altruism* may provide more inconsistency as k increases. In fact, ignoring the degree of dynamic inconsistency might harm, for example, the stability of society. Therefore, inconsistency regulation should be a critical concern for society. In what follows, after presenting the formal definition of a generalized qaiPC, we explore how the intensity of social inconsistency can be characterized through this generalized condition.

We say that consumption streams $\mathbf{z}$ and $\mathbf{z}'$ are k*-periodic* if $z_t = z'_t$ for $t > k$.

k-imperfectly altruistic impartial PC (k-iaiPC): Let $k \in \mathbb{N}$. For any pair of k-periodic consumption streams $\mathbf{z}, \mathbf{z}'$, if $U_{ij}(\mathbf{z}) \geq U_{ij}(\mathbf{z}')$, for all $ij \in \mathcal{I} \times \mathcal{I}$, then $U(\mathbf{z}) \geq U(\mathbf{z}')$.

Consider now a subclass of TSUs satisfying k-iaiPC along with a stationarity-like condition.

Definition 4. A lifetime utility $U : \mathcal{L}^\infty \to \mathbb{R}$ admits a k-*hyperbolic* form if there exist $0 < \beta_1 \leq \ldots \leq \beta_k \leq 1$ and $\delta \in (0,1)$ such that for $\mathbf{z} \in \mathcal{L}^\infty$,

$$U(\mathbf{z}) = u(z_1) + \beta_1 \delta u(z_2) + \beta_1 \beta_2 \delta^2 u(z_3) + \cdots + \prod_{\ell=1}^{k} \beta_\ell \sum_{t=\ell+1}^{\infty} \delta^{t-1} u(z_t). \tag{5}$$

This formulation assumes that the discount factor decreases until period k but is constant thereafter. Parameters β_ℓ can then be assimilated to a measure of the 'horizon $(\ell-1)$'-*bias*. Each β_ℓ can also represent the size of the perceived distance between periods $(\ell-1)$ and ℓ. This definition includes the case of an EDU for $\beta_1 = \cdots = \beta_k = 1$ and that of the classical quasihyperbolic utility for $k = 1$. Note that k-hyperbolic utilities represent a subclass of 'semihyperbolic' utilities, as proposed in Montiel Olea and Strzalecki [2014], for which $\beta_1, \cdots, \beta_k$ are unrestricted. On the other hand, a k-hyperbolic utility requires that $\beta_1, \ldots, \beta_k$ define an increasing sequence, which then reflects the existence of a present bias.

The advantage of considering this class of present-biased utilities is at least twofold. First, any present-biased TSU can be approximated by a k-hyperbolic utility. Therefore, replacing TSUs with this class of utilities does not lead to a loss of generality. Second, this kind of parameterized utility is data friendly. One can apply, for instance, the multiple price list (MPL) to elicit $\beta_1, \ldots, \beta_k$ and, therefore, fully recover the form of the social utility.[17]

k**-Stationarity.** Lifetime utility function U is k-*(delayed) stationary* if, for all $x \in \mathcal{L}$ and all $\mathbf{z}, \mathbf{c}, \hat{\mathbf{c}} \in \mathcal{L}^\infty$,

$$U(\mathbf{z}_k \mathbf{c}) \geq U(\mathbf{z}_k \hat{\mathbf{c}}) \text{ if and only if } U(x, \mathbf{z}_k \mathbf{c}) \geq U(x, \mathbf{z}_k \hat{\mathbf{c}}).$$

This property, k-stationarity, which generalizes classical stationarity, states that if two consumption streams are identical up to period t, then the ranking of these two streams is preserved after adding the same consumption in the current period and delaying both streams one period further. It is clear that a k-hyperbolic utility satisfies k-stationarity, but it is not true that any utility satisfying k-stationarity has a k-hyperbolic form. Delayed stationarity does not impose any restriction on the rate of impatience before period k. Next, it is natural that when k grows, the stationarity-like property becomes stronger. In other words, if $k > \ell$, then k-stationarity implies ℓ-stationarity. Now, we can formally state our result.

[17] The issue of empirical elicitation is beyond the scope of this article. We refer the reader to Montiel Olea and Strzalecki [2014] for work on the MPL and to Cohen et al. [2020] for more general methods.

Theorem 5. k-Stationarity *and* k-iaiPC *hold if and only if there exist nonnegative numbers* α_i *and* γ_i *such that social preferences are represented by* k*-hyperbolic social lifetime utility* U *as in (5) with the following:*

$$u = \sum_i \alpha_i u_i \tag{6}$$

$$\delta = \frac{\sum_j \gamma_j \delta_j^{k+1}}{\sum_j \gamma_j \delta_j^k} \tag{7}$$

$$\beta_\ell = \frac{\sum_j \gamma_j \delta_j^\ell}{\sum_j \gamma_j \delta_j^{\ell-1}} \cdot \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}} \qquad \textit{for all } 1 \leq \ell \leq k. \tag{8}$$

Theorem 5 proves that a society respecting both k-stationarity and $(k+2)$-iaiPC has a social lifetime utility with a k-hyperbolic form. Furthermore, the social instantaneous utility is a weighted average of individual utilities. Additionally, the social discount factor at horizon ℓ *before horizon* k is defined as a proportion of the weighted average of individual discounting function values at horizon ℓ to that at horizon $(\ell - 1)$. The social discount factor at horizon ℓ *after horizon* k is constant and defined as δ, which is a proportion of the weighted average of individual discounting function values at horizon $(k+1)$ to that at horizon k. Therefore, the social discount factor at horizon $\ell \leq k$ can be decomposed into δ and a horizon $(\ell - 1)$-bias denoted by β_ℓ.

In fact, Theorem 5 includes Theorem 4 as a special case of k-iaiPC for $k = 3$. When k goes to infinity, k-stationarity has no bite on stationarity, and k-iaiPC becomes the iPC. Therefore, when $k \to \infty$, Theorem 5 is a special case of Theorem 1 where each individual lifetime utility is an EDU.

Since a social k-hyperbolic lifetime utility displays decreasing impatience, *i.e.*, present bias, it is natural to explore how the degree of decreasing impatience changes when k increases. Let us first provide a notion of comparative present bias.

Definition 5. Lifetime utility U is *more present biased* than lifetime utility V if, for any t, s in $\mathbb{N}$ and $x, y, x', y' \in \mathcal{L}$, $U(x, \bar{z}_*) = U(y_t, \bar{z}_{*-t})$, $V(x', \bar{z}_*) = V(y'_t, \bar{z}_{*-t})$, and $V(x'_s, \bar{z}_{*-s}) \leq V(y'_{t+s}, \bar{z}_{*-\{t+s\}})$ implies $U(x_s, \bar{z}_*) \leq U(y_{t+s}, \bar{z}_{*-\{t+s\}})$.

The intuition behind this definition is as follows:[18] suppose that for lifetime utility U, two streams are equivalently evaluated, one with consumption x at the current time and the other with further consumption y at t. In contrast, another lifetime utility V similarly ranks, *i.e.*, equivalently

[18] Prelec [2004] and Quah and Strulovici [2013] suggest different notions of comparative decreasing impatience but are based on continuous time.

evaluates two other streams, current stream x' and further stream y' at t. Suppose now that all consumption is postponed by time interval s. Whenever utility V prefers further consumption y' in period $(t+s)$ to closer consumption x' in period s, it is always the case that utility U also prefers further consumption y at $(t+s)$ than closer consumption x at s. Since U has earlier preference reversal than V, U is said to be more present biased than V.

Proposition 4. *Let there be nonnegative numbers α_i and γ_j such that $\sum_{i\in\mathcal{I}}\alpha_i = \sum_{j\in\mathcal{I}}\gamma_j = 1$. If $k \leq \hat{k}$, then a society characterized by $(\hat{u}, \hat{\delta}, \{\hat{\beta}_\ell\}_{\ell=1}^{\hat{k}})$ defined as in (6,7,8) is more present biased than a society characterized by $(u, \delta, \{\beta_\ell\}_{\ell=1}^{k})$ defined as in (6,7,8).*

This result indicates that when k increases, the social lifetime utility becomes more present biased. Next, we observe that k-stationarity is necessary to characterize a k-hyperbolic social utility. If we replace k-stationarity with stationarity, society must be a dictatorship.

Corollary 1. *For $k > 2$,* stationarity *and* the k-iaiPC *hold if and only if society is dictatorial.*

As an illustration, consider a society with 2 individuals who have identical instantaneous utilities. For simplicity, assume that the qaiPC holds with $k = 3$. We know that $\big(u(x)+\delta u(y)+\delta^2 u(z)\big)$ is a convex combination of corresponding individual utilities. Then, there exists $\lambda \in [0, 1]$ such that

$$\lambda\delta_1 + (1-\lambda)\delta_2 = d_2 \quad \text{and} \quad \lambda\delta_1^2 + (1-\lambda)\delta_2^2 = d_3.$$

Stationarity requires that $d_2^2 = d_3$. Hence, the only solution must be either $\lambda = 1$ or $\lambda = 0$, which are the two polar cases of dictatorship.

5.4 Perfect or Imperfect Altruism?

In the case of perfect altruism, society would allow for constant discounting when valuing future gains and losses. As we show above, the value of constant discounting lies not in the fact that it simplifies the analysis but in the fact that it satisfies important normative properties such as time consistency and stationarity. It is therefore not surprising that constant discounting systems are widely used by many countries to conduct cost-benefit analysis. However, the shortcomings of constant discounting are not insignificant, especially in regard to environmental policy-making. As pointed out by Karp [2005], Gerlagh and Liski [2017] and many others, such policies are very sensitive to the selection of the discount factor, which may make a society reluctant to accept even moderate costs today to avoid a future disaster.

Consider a society that must choose between two consumption streams:

$$\mathbf{z} = (\underbrace{1, 0, \ldots, 0, -100}_{11 \text{ periods}}, 0, \ldots) \quad \text{and} \quad \mathbf{z}' = (1.1, -0.4, 0, \ldots).$$

The stream $\mathbf{z}$ illustrates that consumption is equal to 1 in the first period, -100 in period 11, and zero the rest of the time. In stream $\mathbf{z}'$, consumption in the first two periods is 1.1 and -0.4, and consumption in the remaining periods is also zero. Suppose society is a constant discounter with $\delta = 0.5$ and has an identity instantaneous utility function. Then, we can see immediately that society will prefer $\mathbf{z}$ to $\mathbf{z}'$:

$$1 - 100 \times 0.5^{10} = 0.9032 > 0.9 = 1.1 - 0.4 \times 0.5.$$

In contrast, if a society is, for example, a quasihyperbolic discounter with present-bias parameter $\beta = 0.8$, society will prefer $\mathbf{z}'$ to $\mathbf{z}$:

$$1 + 0.8 \times 0.5^{10} \times (-100) = 0.9219 < 0.94 = 1.1 + 0.8 \times 0.5 \times (-0.4).$$

In this numerical example, $\mathbf{z}$ can be thought of as a policy that generates a short-run benefit and long-run harm. Alternatively, we can define $\mathbf{z}'$ as a policy that generates a moderate short-term cost to avoid long-term harm. It seems intuitively reasonable for a society to assume responsibility for short-term losses to avoid long-term disasters. However, if this society chooses to be time consistent, it will value future damage as a marginal loss and, therefore, will instead adopt a policy that does not avoid it. In contrast, a society with quasihyperbolic discounting can make a plausible trade-off between benefits and costs in a dynamic setting. Thus, in the example above, quasihyperbolic discounting ameliorates the shortcomings of constant discounting and can then claim to be a normative recommendation for society.

Perfect altruism is certainly a pervasive ethical principle in the minds of policy-makers. However, we argue here that it is not the only possible rational principle. In particular, with respect to environmental policymaking, imperfect altruism seems to us to be more compelling and, above all, more effective at arbitrating between costs and benefits in a dynamic setting.

6 Related Literature

In two different settings, Zuber [2011] and Jackson and Yariv [2014] show that a society with constant discounting that respects the PC cannot aggregate individual lifetime preferences in a nondictatorial manner if individual discount factors and instantaneous utilities are heterogeneous. For our part, although we consider a framework quite similar to Jackson and Yariv's, we show in Theorem 1 that nondictatorial aggregation is possible if social lifetime utility satisfies the iPC, which is weaker than the PC. However, if all individuals reflect EDUs, a linearly separate aggregation rule results in a present-biased social lifetime utility. Since constant discounting is a simple and fruitful assumption for policy-making, we argue in Theorem 3 that if a TSU-type society adheres to a slightly modified iPC, *e.g.*, restricted to pairs of streams differing by only two periods, the society is time consistent and its discount factor is a weighted average of the individual factors.[19] It has been observed by Phelps and Pollak [1968], Barro [1974], Kimball [1987], and Saez-Marti and Weibull [2005] and most recently by Galperti and Strulovici [2017] that altruism toward immediate generation would lead to time consistency. Although our problem and framework are substantially different from theirs, the fundamental idea we defend and prove is that such time consistency can be derived from altruism between two arbitrary, *i.e.*, not necessarily consecutive, generations.

Chambers and Echenique [2018] ignore the heterogeneity of individual instantaneous utilities and suggest three rules for aggregating discount factors. One of them proposes aggregation by means of a weighted average method, which can then be seen as an alternative approach to a special case of our Theorem 3. However, because of the significant difference between the two contexts, the views conveyed by the respective social principles are fundamentally different. In contrast, Feng and Ke [2018] argue that the preferences of successive generations must be accounted for. Therefore, these authors suggest an intergenerational PC and characterize a constant social discount factor that is greater than any individual factor.[20] Chichilnisky, Hammond, and Stern [2020] consider that future generations are threatened with extinction and thus propose a social discounting of 'extinction.' There are many other approaches to studying social time consistency. Millner and Heal [2018] demonstrates that a society can be time consistent if one abandons the assumption that social consumption is invariant. The same exercise, but in a continuous time framework, is performed by Drouhin [2020]. More recently, Hayashi and Lombardi [2021] develops a consensual PC for a context wherein both individuals and society are of the EDU type, which gives rise to a utilitarian social instantaneous utility and a dictatorial social constant discount factor.

Although constant social updating is an irresistible form, it is nevertheless rarely observed in policy-making. In fact, the institutional rotation of political power (see, Harstad [2020]) or the cost

[19] This result can be thought of as an axiomatization of the exponential social discounter. The first axiomatization of this type of discounting behavior is due to Koopmans [1960]. It was then extended by Fishburn and Rubinstein [1982] and Bleichrodt, Rohde, and Wakker [2008] in a different way.

[20] Drugeon and Wignolle [2020] study a similar collective decision problem studied by assuming hyperbolically discounted individuals.

of commitment (see, Laibson [2015]) would be responsible for triggering present-biased policies. To understand the underlying behavioral mechanism involved, we show in Theorem 4 that if social lifetime utility satisfies the iPC for any pair of consumption streams that are different in the first 3 periods, *i.e.*, the qaiPC, then with quasi-stationarity, this social lifetime utility admits a form of quasihyperbolic discounting as shown in Phelps and Pollak [1968] and Laibson [2015] and many others.[21] In fact, Gollier and Zeckhauser [2005] and Jackson and Yariv [2014] show that a social lifetime utility satisfying the PC is present biased if individuals have heterogeneous discount factors. Therefore, our theorem 5 generalizes this observation by showing that different versions of the iPC can characterize comparatively different levels of present bias. This result is related to Millner [2020], who argue that a society subjected to various normative arguments is weakened. Theorem 5 can then be interpreted as an axiomatic evaluation of these multiple social principles. In contrast to the present-bias approach, recent papers by Gonzalez, Lazkano, and Smulders [2018] and Ray [2018] show that a society can exhibit future bias if there is a conflict of interest between future generations.

To the best of our knowledge, we are the first to identify the social principles that are necessary to obtain discount factors and instantaneous utilities to aggregate distinctly. As noted by Jackson and Yariv [2015], a TSU is quite analogous to a subjective expected utility. Indeed, it is possible to interpret time as a continuous set of states and the discount function as a probability distribution defined over this set. In this respect, our result, while restricting the PC to avoid the impossibilities highlighted by Jackson-Yariv and Zuber while obtaining a separate aggregation, is conceptually related to Gilboa, Samet, and Schmeidler [2004] and Billot and Qu [2021], who show that a restricted PC allows for separate aggregation of heterogeneous beliefs and tastes. However, while the results are conceptually similar, the problems are radically different.

Weitzman [2001] point to the unquestionable and substantial reality of individual heterogeneity, that is, the natural disagreement between individuals' feelings or even experts' opinions about health issues, a disagreement that relates as well to the trade-offs between present and future benefits from a social point of view. However, Weitzman [2001] and Freeman and Groom [2014] apply the 'gamma discounting method' to form the social discount factor, which is a different method from the one we propose. Indeed, this has been known since Harsanyi [1955] and more recently, Zuber [2011] showed that a society respecting the PC cannot make decisions in the same way as individuals when they are heterogeneous in their lifetime preferences, whether this comes from heterogeneity in discount factors, instantaneous utilities or both. We show, however, that a so-

[21]This result can be seen as an axiomatization of the quasihyperbolic social discounter. The axiomatization of this type of behavior is also found in Hayashi [2003], Attema, Bleichrodt, Rohde, and Wakker [2010], Noor [2009] and Montiel Olea and Strzalecki [2014].

ciety that respects another type of Paretian unanimity, namely, the so-called iPC, behaves in the same way as individuals when individuals' preferences are separable over time. Furthermore, we show that the social discount function and instantaneous utility function correspond to a weighted average of the individual discount functions and individual utilities, respectively.

7 CONCLUSION

The principle underlying the PC is not always convincing, especially when individuals are heterogeneous in both their discounting functions and their instantaneous utilities. We point out the limitations of the Paretian principle for a heterogeneous context and provide the conditions under which a society should follow a separate aggregation rule and admit a time-consistent or time-inconsistent representation for social preferences.

Although we argue that the plausibility of time consistency depends mostly on the problem at hand, the literature shows that neither social time consistency nor inconsistency prevails in empirical or experimental studies. While Adams, Cherchye, De Rock, and Verriest [2014] show that household consumption behavior can be described by time-consistent preferences, Jackson and Yariv [2014] find a social violation of time consistency. Therefore, practically, it is possible to test social decision-making by distinguishing which of our principles applies.

APPENDIX

A PRELIMINARY RESULTS

We first prove a general aggregation result, *i.e.*, Proposition A1 below. One may find different versions of this proof, but we choose to present this result for two reasons. On the one hand, it is expressed in our setting, and on the other hand, it will be used repeatedly in the following proofs. To avoid tiresome duplication, this result is singled out at the beginning.

If $k \in \mathbb{N}$, for $ij \in \mathcal{I} \times \mathcal{I}$, let $U_{ij}^k : \mathcal{L}^k \to \mathbb{R}$ and $U^k : \mathcal{L}^k \to \mathbb{R}$ be two real-valued functions defined on $\mathcal{L}^k$ with a convex range. Consider now a unanimity postulate.

k-Unanimity: Fix $k \in \mathbb{N}$. For any $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^k$, if $U_{ij}^k(\mathbf{z}) \geq U_{ij}^k(\mathbf{z}')$, for all $i, j \in \mathcal{I} \times \mathcal{I}$, then $U^k(\mathbf{z}) \geq U^k(\mathbf{z}')$. Furthermore, if there exists fictitious ij such that if $U_{ij}^k(\mathbf{z}) > U_{ij}^k(\mathbf{z}')$, then $U^k(\mathbf{z}) > U^k(\mathbf{z}')$.

Proposition A1. *Under* MAC,[22] k-Unanimity *holds if and only if there exist positive numbers* λ_{ij} *and real number* μ *such that for* $\mathbf{z} \in \mathcal{L}^k$*, the following applies:*

$$U^k(\mathbf{z}) = \sum_{ij \in \mathcal{I} \times \mathcal{I}} \lambda_{ij} U^k_{ij}(\mathbf{z}) + \mu.$$

Proof. Let

$$Y = \{\mathbf{y} \in \mathbb{R}^{n^2+1} | y_0 \leq -1 \text{ and } y_{ij} \geq 0 \text{ for } ij \in \mathcal{I} \times \mathcal{I}\},$$

and

$$A = \Big\{ \big(U^k(\mathbf{z}) - U^k(\hat{\mathbf{z}}), U^k_{11}(\mathbf{z}) - U^k_{11}(\hat{\mathbf{z}}), \ldots, U^k_{nn}(\mathbf{z}) - U^k_{nn}(\hat{\mathbf{z}})\big) \in \mathbb{R}^{n^2+1} \Big| \mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^k \Big\}.$$

Since $\mathcal{L} = \Delta(X)$, set $\{(U^k(\mathbf{z}), U^k_{11}(\mathbf{z}), \ldots, U^k_{nn}(\mathbf{z})) \big| \mathbf{z} \in \mathcal{L}\}$ is convex. Therefore, A is convex and symmetric with respect to vector $\mathbf{0}$. According to k-Unanimity, we have $Y \cap A = \emptyset$. Now, define the vector space spanned by A:

$$\text{span}(A) = \Big\{ \sum_{\ell=1}^{m} r_\ell \mathbf{a}_\ell \Big| m \in \mathbb{N}, r_\ell \in \mathbb{R} \text{ and } \mathbf{a}_\ell \in A \Big\}.$$

It is immediately clear that $Y \cap \text{span}(A) = \emptyset$. Since Y and $\text{span}(A)$ are polyhedral, nonempty and mutually disjointed, the strictly separating theorem (see, *e.g.*, Rockafellar, Corollary 19.3.3) means that there exist $\boldsymbol{\pi} = (\pi, \pi_{11}, \ldots, \pi_{nn})$ such that for all $\mathbf{a} \in \text{span}(A)$ and $\mathbf{y} \in Y$, $\boldsymbol{\pi} \cdot \mathbf{y} > \boldsymbol{\pi} \cdot \mathbf{a}$. Note that for all $\mathbf{a} \in \text{span}(A)$, we have $\boldsymbol{\pi} \cdot \mathbf{a} = 0$. (Suppose the opposite. Then, there must exist $\hat{\mathbf{a}} \in \text{span}(A)$ such that $\boldsymbol{\pi} \cdot \hat{\mathbf{a}} > 0$, and this is wlog since set A is symmetric. Therefore, there exists a large enough $r \in \mathbb{R}$ such that $\boldsymbol{\pi} \cdot r\hat{\mathbf{a}} > \boldsymbol{\pi} \cdot \mathbf{y}$, which is a contradiction.)

Select $\mathbf{y} = (-1, 0, \ldots, 0) \in Y$. The above inequality, *i.e.*, $\boldsymbol{\pi} \cdot \mathbf{y} > 0$, implies that $-\pi > 0$; that is, $\pi < 0$. Thus, for all $\mathbf{z}, \hat{\mathbf{z}} \in \mathcal{L}^k$,

$$U^k(\mathbf{z}) - U^k(\hat{\mathbf{z}}) = \sum_{ij \in \mathcal{I} \times \mathcal{I}} \frac{\pi_{ij}}{-\pi} \big[U^k_{ij}(\mathbf{z}) - U^k_{ij}(\hat{\mathbf{z}}) \big].$$

Fix $\hat{\mathbf{z}}$. For $ij \in \mathcal{I} \times \mathcal{I}$, define λ_{ij} as $\frac{\pi_{ij}}{-\pi}$. Define μ as $U^k(\hat{\mathbf{z}}) - \sum_{ij \in \mathcal{I} \times \mathcal{I}} \lambda_{ij} U^k_{ij}(\hat{\mathbf{z}})$. Therefore, for all $\mathbf{z} \in \mathcal{L}^k$, we have $U^k(\mathbf{z}) = \sum_{ij} \lambda_{ij} U^k_{ij}(\mathbf{z})$.

To verify that each λ_{ij} is positive, let $\mathbf{y}$ be such that $y = -1, y_{ij} = r > 0$ and $y_{i'j'} = 0$ for $ij \neq i'j'$. The existence of $\mathbf{y}$ is guaranteed by the MAC. Therefore, $\boldsymbol{\pi} \cdot \mathbf{y} > 0$ implies $-\pi + r\pi_{ij} > 0$.

[22] Recall the MAC: there exist $x^*, x_* \in \mathcal{L}$ such that for all $k \in \mathbb{N}$ and all $ij \in \mathcal{I} \times \mathcal{I}$, $U_{ij}(x^*) > U_{ij}(x_*)$.

Hence, $\pi_{ij} > 0$, for ij, which then denotes that each $\lambda_{ij} > 0$. □

B Proof of Section 3: Proposition 1

The necessity part whereby a dictatorial society satisfies PC is straightforward. We only prove the sufficiency part.

Note that if we assume that $k = \infty$ and $U_{ij}(\mathbf{z}) = U_i(\mathbf{z}) = \sum_t d_{it}u_i(z_t)$, for all $j \in \mathcal{I}$, then the PC is equivalent to k-Unanimity for $k = \infty$. Since the PC is satisfied, by Proposition A1, for $k = \infty$, there exist nonnegative $\{\lambda_i\}_{i\in\mathcal{I}}$ such that for $\mathbf{z} \in \mathcal{L}^\infty$, the following applies:

$$\sum_t d_t u(z_t) = \sum_{i\in\mathcal{I}} \lambda_i \sum_t d_{it}u_i(z_t) + \mu.$$

By normalization, $u_i(x_*) = 0$ and $u_i(x^*) = 1$ for all i. First, take $\mathbf{z} = (x_*, \ldots, x_*)$. Then, we have $\mu = 0$. Second, take $\mathbf{z} = (x^*, x_*, x_*, \ldots)$. Then, this implies that $\sum_{i\in\mathcal{I}} \lambda_i = 1$. Therefore, for all $t \in \mathbb{N}$, $d_t = \sum_{i\in\mathcal{I}} \lambda_i d_{it}$. Additionally, the Harsanyi aggregation theorem on $\mathcal{L} = \Delta(X)$ requires that for all $z \in \mathcal{L}$, $u(z) = \sum_{i\in\mathcal{I}} \lambda_i u_i(z)$. Hence, this requires that $\sum \lambda_i d_{it} u_i(z) = \sum \lambda_i d_{it} \cdot \sum \lambda_i u_i(z)$, *i.e.*:

$$\sum_{i\in\mathcal{I}} \lambda_i \big(d_{it} - \sum_{i\in\mathcal{I}} \lambda_i d_{it}\big) u_i(z) = 0.$$

By regularity, we know that $\{u_i\}_{i\in\mathcal{I}}$ are independent. Thus, for every $i \in \mathcal{I}$, $d_{it} - \sum \lambda_i d_{it} = 0$. Hence, $\lambda_i = 0$ or 1.

C Proofs of Section 4

C.1 Proof of Theorem 1

Necessity part. Suppose that for $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$, all $ij \in \mathcal{I} \times \mathcal{I}$, $U_{ij}(\mathbf{z}) \geq U_{ij}(\mathbf{z}')$. Since each α_i and γ_j are nonnegative, we have $\alpha_i\gamma_j U_{ij}(\mathbf{z}) \geq \alpha_i\gamma_j U_{ij}(\mathbf{z}')$ for all $ij \in \mathcal{I} \times \mathcal{I}$. Now, $U = \sum_{ij} \alpha_i\gamma_j U_{ij}$ implies $U(\mathbf{z}) \geq U(\mathbf{z}')$, which proves the iPC.

Sufficiency part. Suppose that the iPC is satisfied. The iPC is equivalent to k-Unanimity for $k = \infty$, where $U_{ij}(\mathbf{z}) = \sum_t d_{jt}u_i(z_t)$ for all $\mathbf{z} \in \mathcal{L}^\infty$. Then, according to Proposition A1, there exist nonnegative $\{\lambda_{ij}\}_{ij\in\mathcal{I}\times\mathcal{I}}$ such that for $\mathbf{z} \in \mathcal{L}^\infty$, $U(\mathbf{z}) = \sum_{ij} \lambda_{ij} U_{ij}(\mathbf{z})$, *i.e.*:

$$\sum_{t=1}^{\infty} d_t u(z_t) = \sum_{ij} \lambda_{ij} \sum_{t=1}^{\infty} d_{jt} u_i(z_t).$$

Recall that, by normalization, $u_i(x_*) = 0$ and $u_i(x^*) = 1$ for all i. Accordingly, defining $\mathbf{z}$ as $z_1 = x^*$ and $z_t = x_*$, for $t \neq 1$ implies $\sum_{ij} \lambda_{ij} = 1$. Let $\alpha_i = \sum_j \lambda_{ij}$ and $\gamma_j = \sum_i \lambda_{ij}$. Then, for $z \in \mathcal{L}$, $u(z) = \alpha_i u_i(z)$. For $t \in \mathbb{N}$, consider stream $\mathbf{z}$ such that $z_t = x^*$ and $z_s = x_*$ for $s \neq t$. Therefore, $d_t = \sum_j \gamma_j d_{jt}$. Hence, U separately aggregates instantaneous individual utilities and individual discount functions.

C.2 Proof of Theorem 2

Since the necessity part is straightforward, we only show the sufficiency part.

Suppose that the cvPC is satisfied.

First, consider the cvPC restricted to pairs of constant common-valued consumption streams. Then, it is equivalent to the following: for all $x, y \in \mathcal{L}$, if $u_i(x) \geq u_i(y)$ and for all $i \in \mathcal{I}$, then $u(x) \geq u(y)$. Clearly, this is the PC of a lottery context. Therefore, Harsanyi's Aggregation Theorem implies that there exist nonnegative numbers α_i with $\sum_{i \in \mathcal{I}} \alpha_i = 1$ such that $u(x) = \sum_i \alpha_i u_i(x)$ for $x \in \mathcal{L}$.

Second, consider the cvPC restricted to pairs of nonconstant common-valued consumption streams. According to the MAC, $u_i(x^*) = 1$ and $u_i(x_*) = 0$ for all $i \in \mathcal{I}$. Since each individual has an expected utility representation over $\mathcal{L}$, for all $z \in \Delta(\{x^*, x_*\})$, $u_i(z) = u_j(z)$ for all $i, j \in \mathcal{I}$. The set of common-valued streams is defined by the following:

$$\mathcal{L}^\infty_{\text{cv}} = \{\mathbf{z} \in \mathcal{L}^\infty : z_t \in \Delta(\{x^*, x_*\}) \text{ for all } t \in \mathbb{N}\}.$$

It is immediately apparent that $\mathcal{L}^\infty_{\text{cv}}$ is convex, and therefore, set $\{(U(\mathcal{L}^\infty_{\text{cv}}), U_1(\mathcal{L}^\infty_{\text{cv}}), \ldots, U_n(\mathcal{L}^\infty_{\text{cv}})\}$ is also convex. Applying the separation theorem, there is a nonnegative γ_i with $\sum_i \gamma_i = 1$ such that $U(\mathbf{z}) = \sum_i \gamma_i U_i(\mathbf{z})$ for all $\mathbf{z} \in \mathcal{L}^\infty_{\text{cv}}$. Let $\mathbf{z} = (x_*, x^*, x_*, x_*, \ldots)$. We replace this with the expression and obtain $d_2 = \sum_i \gamma_i d_{i2}$. Similarly, we can replace $(x_*, x_*, x^*, x_*, \ldots)$ in the expression and obtain $d_3 = \sum_i \gamma_i d_{i3}$. By repeating the process, we have $d = \sum_i \gamma_i d_i$. Hence, the social discount function is a convex combination of individual discount functions.

C.3 Proof of Lemma 1

Necessity part. Suppose that $U(x_t, \bar{z}_{*-t}) = U(y_s, \bar{z}_{*-s})$, where $t > s$. This implies that for $k > 0$, the following applies:

$$\begin{aligned}\frac{u(y)}{u(x)} &= \frac{d_t}{d_s} = \frac{d_t}{d_{t-1}} \times \frac{d_{t-1}}{d_{t-2}} \times \cdots \times \frac{d_{s+1}}{d_s} \\ &= \delta(t) \times \delta(t-1) \times \cdots \times \delta(s+1) \\ &\leq \delta(t+k) \times \delta(t+k-1) \times \cdots \times \delta(s+k+1) \\ &= \frac{d_{t+k}}{d_{s+k}}.\end{aligned}$$

Therefore, it is clear that $U(x_{t+k}, \bar{z}_{*-(t+k)}) \geq U(y_{s+k}, \bar{z}_{*-(s+k)})$.
Sufficiency part. Since u is continuous and has a range containing $[0, 1]$, for $t > 1$, there exist $x, y \in \mathcal{L}$ such that

$$\delta_d(t) = \frac{d_t}{d_{t-1}} = \frac{u(x)}{u(y)}.$$

Then, present bias implies that for $k \in \mathbb{N}$, $d_{t+k}u(x) \geq d_{t+k-1}u(y)$. Therefore, we have

$$\frac{d_t}{d_{t-1}} \geq \frac{d_t}{d_{t-1}}.$$

Since this expression is valid for all t, the discount factor $\delta_d(t)$ is increasing in t.

A similar process can be easily applied to prove the case of constant impatience.

C.4 Proof of Proposition 2

From Lemma 1, it suffices to show that $\delta_d(t)$ is increasing. Let $\delta_d^i(t)$ be the discount factor of individual i at horizon t. Then,

$$\begin{aligned}\delta_d(t+1) = \frac{d_{t+1}}{d_t} &= \frac{\sum \gamma_i d_{i(t+1)}}{\sum \gamma_i d_{it}} \\ &= \frac{\sum \gamma_i \delta_d^i(t+1) d_{it}}{\sum \gamma_i d_{it}}.\end{aligned}$$

Similarly, we have the following:

$$\delta_d(t) = \frac{\sum \gamma_i d_{it}}{\sum \frac{\gamma_i}{\delta_d^i(t)} d_{it}}.$$

Since $\delta_d^i(t)$ is increasing, for all i, it follows therefrom that

$$\frac{\delta_d^j(t+1)}{\delta_d^i(t)} + \frac{\delta_d^i(t+1)}{\delta_d^j(t)} \geq 2\sqrt{\frac{\delta_d^j(t+1)}{\delta_d^i(t)} \times \frac{\delta_d^i(t+1)}{\delta_d^j(t)}} \geq 2.$$

Now, since coefficients γ_i are nonnegative, it is also true that

$$\Big(\frac{\delta_d^j(t+1)}{\delta_d^i(t)} + \frac{\delta_d^i(t+1)}{\delta_d^j(t)}\Big)\gamma_i\gamma_j \geq 2\gamma_i\gamma_j \quad \text{for all } i,j \in \mathcal{I}.$$

Therefore,

$$\Big(\sum \gamma_i \delta_d^i(t+1) d_{it}\Big) \times \Big(\sum \frac{\gamma_i}{\delta_d^i(t)} d_{it}\Big) \geq \Big(\sum \gamma_i d_{it}\Big)^2,$$

which in turn implies that $\delta_d(t+1) \geq \delta_d(t)$.

D PROOFS OF SECTION 5

D.1 Proof of Theorem 3

The necessity part is straightforward; therefore, we only prove the sufficiency part.

Suppose that the paiPC and stationarity hold.

(i) We have to show that social lifetime utility U is an EDU. For this, it is sufficient to demonstrate that U satisfies all postulates of Koopmans [1960]. First, Postulate 1 is implied by the continuity of U. Next, Postulates 3 and 3′ are implied by the time separability of U. Moreover, stationarity corresponds to Postulate 4. Since $u_i(x^*) > u_i(x_*)$, for all $i \in \mathcal{I}$, the paiPC implies that $u(x^*) > u(x_*)$. Therefore, by time additivity, for $\mathbf{z} \in \mathcal{L}^\infty$, $U(x^*, \mathbf{z}) > U(x_*, \mathbf{z})$, which implies Postulate 2. Finally, since $u(x^*) \geq u(z) \geq u(x_*)$, for all $z \in \mathcal{L}$ and all $\mathbf{z} \in \mathcal{L}^\infty$, we have $U(x^*, \ldots, x^*) \geq U(\mathbf{z}) \geq U(x_*, \ldots, x_*)$, that is, Postulate 5. Hence, by Koopmans' Theorem, social lifetime utility U is an EDU.

(ii) For $k = 2$, paiPC is equivalent to 2-Unanimity, where $U_{ij}^2(x,y) = u_i(x) + \delta_j u_i(y)$ and $U^2(x,y) = u(x) + \delta u(y)$ for all $(x,y) \in \mathcal{L}^2$. Therefore, Proposition A1 implies that there exist nonnegative λ_{ij} and a real number μ such that $U^2 = \lambda_{ij} U_{ij}^2 + \mu$, that is, for any $(x,y) \in \mathcal{L}^2$:

$$u(x) + \delta u(y) = \sum_{ij} \lambda_{ij} u_i(x) + \sum_{ij} \lambda_{ij}\delta_j u_i(y) + \mu.$$

Let $x = y = x_*$. Then, $u(x_*) = u_i(x_*) = 0$ implies $\mu = 0$. Now, take $y = x_*$, and let $\alpha_i = \sum_j \lambda_{ij}$. The above equation becomes $u(x) = \sum_i \alpha_i u_i(x)$,, which proves that social instantaneous utility

u is a convex combination of individual instantaneous utilities. Suppose that $x = x_*$ and $y = x^*$. Then, $u(x^*) = u_i(x^*) = 1$ for all i. Let $\gamma_j = \sum_i \lambda_{ij}$. The above equation then becomes $\delta = \sum_j \gamma_j \delta_j$, which proves that social discount factor δ is a convex combination of individual discount factors.

D.2 *Proof of Proposition 3*

Necessity part. Stationarity is immediate. We have only proven 2-paiPC. We fix $k, m \in \mathbb{N}$. Suppose that $U_{ij}(x_k, y_{k+m}, \mathbf{z}_{-(k,k+m)}) \geq U_{ij}(x'_k, y'_{k+m}, \mathbf{z}_{-(k,k+m)})$ for each $ij \in \mathcal{I} \times \mathcal{I}$. Since U is an EDU, this is equivalent to $u_i(x) + \delta_j^m u_i(y) \geq u_i(x') + \delta_j^m u_i(y')$. We know that $u = \sum_{i \in \mathcal{I}} \alpha_i u_i$ and $\delta = \sum_{j \in \mathcal{I}} \gamma_j \delta_j$ with both α_i and γ_j being nonnegative with $\sum_i \alpha_i = \sum_j \gamma_j = 1$. Therefore, there exists a nonnegative $\hat{\gamma}_j$ with $\sum_j \hat{\gamma}_j = 1$ such that $\delta^m = \sum_j \hat{\gamma}_j \delta_j^m$. Hence,

$$\sum_i \hat{\gamma}_j \sum_i \alpha_i \big(u_i(x) + \delta_j^m u_i(y)\big) \geq \sum_i \hat{\gamma}_j \sum_i \alpha_i \big(u_i(x') + \delta_j^m u_i(y')\big),$$

i.e., $u(x) + \delta^m u(y) \geq u(x') + \delta^m u(y')$. Thus, we have

$$U(x_k, y_{k+m}, \mathbf{z}_{-(k,k+m)}) \geq U(x'_k, y'_{k+m}, \mathbf{z}_{-(k,k+m)}).$$

Sufficiency part. Suppose that stationarity and the 2-paiPC hold. From a similar argument as that used in the proof of Theorem 3, we know that social lifetime utility U is an EDU (u, δ). Therefore, the 2-paiPC is equivalent to k-Unanimity for $k = 2$ and $U_{ij}^2 = u_i + \delta_j^m u_i$. According to Proposition A1, there exist nonnegative λ_{ij} and μ such that for $x, y \in \mathcal{L}$, $u(x) + \delta^m u(y) = \sum_{ij} \lambda_{ij} \big(u_i(x) + \delta_j^m u_i(y)\big) + \mu$. Furthermore, $\mu = 0$ and $\sum_{ij} \lambda_{ij} = 1$. Moreover, let $\alpha_i = \sum_j \lambda_{ij}$ and $\gamma_j = \sum_i \lambda_{ij}$. We know that $u = \sum_i \alpha_i u_i$ and $\delta^m = \sum_j \gamma_j \delta_j^m$. Furthermore, since $\delta^m \in [\min_j \delta_j^m, \max_j \delta_j^m]$, this means that $\delta \in [\min_j \delta_j, \max_j \delta_j]$. Hence, there exist nonnegative $\hat{\gamma}_j$ with $\sum_j \hat{\gamma}_j = 1$ such that $\delta = \sum_j \hat{\gamma}_j \delta_j$.

D.3 *Proof of Theorem 5*

If $k \in \mathbb{N}$, we define function $U^k : \mathcal{L}^\infty \to \mathbb{R}$ for $\mathbf{z} \in \mathcal{L}^\infty$ as follows:

$$U^k(\mathbf{z}) = U(x_*, \ldots, x_*, \mathbf{z}) = \sum_{t=1}^{\infty} d_{t+k} u(z_t).$$

We want first to show that this so-defined function U^k satisfies all five postulates of Koopmans [1960] and, therefore, is an EDU.

(i) Postulate 1 follows from the definition of U^k and continuity of u.

(ii) Since $u_i(x^*) > u_i(x_*)$, for all $i \in \mathcal{I}$, k-Unanimity implies that $u(x^*) > u(x_*)$. Therefore, due to the time additivity of U^k and $d_{k+1} > 0$, for $\mathbf{z} \in \mathcal{L}^\infty$, $U^k(x^*, \mathbf{z}) > U^k(x_*, \mathbf{z})$, which implies Postulate 2.

(iii) Postulate 3 follows immediately from the time additivity of U^k. That is, for all $x, y \in \mathcal{L}$ and $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$, the following applies:

$$U^k(x, \mathbf{z}) \geq U^k(y, \mathbf{z}) \Leftrightarrow u(x) \geq u(y) \Leftrightarrow U^k(x, \mathbf{z}') \geq U^k(y, \mathbf{z}').$$

Similarly, we have the following:

$$U^k(x, \mathbf{z}) \geq U^k(x, \mathbf{z}') \Leftrightarrow \sum_{t=2}^{\infty} u(z_t) \geq \sum_{t=2}^{\infty} u(z'_t) \Leftrightarrow U^k(y, \mathbf{z}) \geq U^k(y, \mathbf{z}').$$

(iv) Let $x \in \mathcal{L}$ and $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$, such that

$$\begin{aligned} U^k(\mathbf{z}) \geq U^k(\mathbf{z}') &\Leftrightarrow U(x_*, \ldots, x_*, \mathbf{z}) \geq U(x_*, \ldots, x_*, \mathbf{z}') \\ &\Leftrightarrow U(x_*, \ldots, x, \mathbf{z}) \geq U(x_*, \ldots, x, \mathbf{z}') \\ &\Leftrightarrow U(x_*, \ldots, x, \mathbf{z}) \geq U(x_*, \ldots, x, \mathbf{z}') \\ &\Leftrightarrow U^k(x, \mathbf{z}) \geq U^k(x, \mathbf{z}'). \end{aligned}$$

The first and last equivalence relations are given by definition. The second equivalence stems from the time additivity of U. The third equivalence is induced by the property of k-stationarity and proves that Postulate 4 holds.

(v) Note that $u(x^*) > u(x_*)$. Therefore, for two streams $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$ such that $z_t = x^*$ and $z'_t = x_*$, for all $t \in \mathbb{N}$, because d_t is positive, for any $\hat{\mathbf{z}} \in \mathcal{L}^\infty$, we have the following:

$$U^k(\mathbf{z}) \geq U^k(\hat{\mathbf{z}}) \geq U^k(\mathbf{z}'),$$

which demonstrates Postulate 5.

(vi) Finally, we have to demonstrate Postulate $3'$. Let $x, y, x', y' \in \mathcal{L}$ and $\mathbf{z}, \mathbf{z}' \in \mathcal{L}^\infty$. Hence,

$$\begin{aligned} U^k(x, y, \mathbf{z}) \geq U^k(x', y', \mathbf{z}) &\Leftrightarrow d_{k+1}u(x) + d_{k+2}u(y) \geq d_{k+1}u(x') + d_{k+2}u(y') \\ &\Leftrightarrow U^k(x, y, \mathbf{z}') \geq U^k(x', y', \mathbf{z}'). \end{aligned}$$

Similarly,

$$U^k(x,y,\mathbf{z}) \geq U^k(x',y,\mathbf{z}') \Leftrightarrow d_{k+1}u(x) + \sum_{t=2}^{\infty} d_{t+2}u(z_t) \geq d_{k+1}u(x') + \sum_{t=2}^{\infty} d_{t+2}u(z'_t)$$
$$\Leftrightarrow U^k(x,y',\mathbf{z}) \geq U^k(x',y',\mathbf{z}').$$

Therefore, U^k defined on $\mathcal{L}^\infty$ satisfies Postulates 1-5 and $3'$. Then, according to Koopmans' Theorem, there exist $a \in (0,1)$ and continuous function u on $\mathcal{L}$ such that

$$U^k(\mathbf{z}) = \sum_{t=1}^{\infty} a^{t-1}u(z_t).$$

Since the representation is unique, there exists $b > 0$ such that for $\mathbf{z} \in \mathcal{L}^\infty$,

$$U(\mathbf{z}) = \sum_{t=1}^{k} d_t u(z_t) + b \sum_{t=k+1}^{\infty} a^{t-k-1}u(z_t).$$

For $\mathbf{z} \in \mathcal{L}^\infty$, we write the following:

$$U^{k+2}(\mathbf{z}) = \sum_{t=1}^{k} d_t u(z_t) + bu(z_{k+1}) + bau(z_{k+2}).$$

Therefore, $(k+2)$-Unanimity can be equivalently written as follows: for any $z_1, \dots, z_{k+2}$ and $z'_1, \dots, z'_{k+2}$ in $\mathcal{L}$,

$$U_{ij}^{k+2}(\mathbf{z}) \geq U_{ij}^{k+2}(\mathbf{z}'), \text{ for all } i,j \in \mathcal{I} \Longrightarrow U^{k+2}(\mathbf{z}) \geq U^{k+2}(\mathbf{z}').$$

Hence, there exist nonnegative λ_{ij} such that

$$U^{k+2}(\mathbf{z}) = \sum_{ij} \lambda_{ij} U_{ij}^{k+2}(\mathbf{z}). \tag{9}$$

Let $\mathbf{z}$ be such that $z_1 = x^*$ and $z_t = x_*$ for $t \neq 1$. Then, (9) implies $\sum_{ij} \lambda_{ij} = 1$. For $i \in \mathcal{I}$, denote $\alpha_i = \sum_j \lambda_{ij}$. Therefore, for all $z \in \mathcal{L}$, $u(z) = \sum_i \alpha_i u_i(z)$, which proves that u is a convex combination of u_i.

For $j \in \mathcal{I}$, denote $\gamma_j = \sum_i \lambda_{ij}$. Clearly, $\sum_j \gamma_j = 1$. Now, let $\mathbf{z}, \mathbf{z}'$ be such that $z_{k+1} = x^*$ and $z_t = x_*$ for $z \neq k+1$, and $z'_{k+2} = x^*$ and $z_t = x_*$ for $z \neq k+2$. Substituting $\mathbf{z}$ and $\mathbf{z}'$ into (9) implies $b = \sum_j \gamma_j \delta_j^k$ and $ba = \sum_j \gamma_j \delta_j^{k+1}$. Define $\delta = a$ as follows:

$$\delta := a = \frac{\sum_j \gamma_j \delta_j^{k+1}}{\sum_j \gamma_j \delta_j^k}. \tag{10}$$

Let $\overline{\delta} = \max_j \delta_j$ and $\underline{\delta} = \min_j \delta_j$. Therefore, since γ_j is nonnegative, for any j,

$$\frac{\sum_j \gamma_j \delta_j^k \underline{\delta}}{\sum_j \gamma_j \delta_j^k} \leq \frac{\sum_j \gamma_j \delta_j^{k+1}}{\sum_j \gamma_j \delta_j^k} \leq \frac{\sum_j \gamma_j \delta_j^k \overline{\delta}}{\sum_j \gamma_j \delta_j^k},$$

and hence $\underline{\delta} \leq \delta \leq \overline{\delta}$.
Let $\hat{b}$ be such that $\hat{b} \cdot a^k = b$. Therefore,

$$\hat{b} = \frac{(\sum_j \gamma_j \delta_j^k)^{k+1}}{(\sum_j \gamma_j \delta_j^{k+1})^k}.$$

Let $\mathbf{z}$ be such that $z_k = x^*$ and $z_t = x_*$ for $t \neq k$. Substituting $\mathbf{z}$ into (9) implies $d_k = \sum_j \gamma_j \delta_j^{k-1}$. Similarly, for all $\ell = 2, \ldots, k-1$, $d_\ell = \sum_j \gamma_j \delta_j^{\ell-1}$. Now, define $\beta_k, \ldots, \beta_1$ recursively as follows:

$$\begin{aligned}
\beta_k &= \hat{b} \times \frac{\delta^{k-1}}{d_k} \\
\beta_{k-1} &= \frac{\hat{b}}{\beta_k} \times \frac{\delta^{k-2}}{d_{k-1}} \\
&\vdots \\
\beta_\ell &= \frac{\hat{b}}{\beta_k \beta_{k-1} \cdots \beta_{\ell+1}} \times \frac{\delta^{\ell-1}}{d_\ell} \\
&\vdots \\
\beta_1 &= \frac{\hat{b}}{\beta_k \beta_{k-1} \cdots \beta_2}.
\end{aligned}$$

Hence, for every $\ell = 2, \ldots, k$, $d_\ell = \beta_1 \beta_2 \cdots \beta_{\ell-1} \delta^{\ell-1}$. Thus, for $\mathbf{z} \in \mathcal{L}^\infty$, U should take the following form:

$$U(\mathbf{z}) = u(z_1) + \beta_1 \delta u(z_2) + \beta_1 \beta_2 \delta^2 u(z_3) + \cdots + \prod_{j=1}^{k} \beta_j \sum_{t=j+1}^{\infty} \delta^{t-1} u(z_t),$$

in which δ is given by (10). Furthermore, substituting $\hat{b}$ and each d_ℓ into the above expression implies the following:

$$\beta_k = \frac{(\sum_j \gamma_j \delta_j^k)^2}{(\sum_j \gamma_j \delta_j^{k-1})(\sum_j \gamma_j \delta_j^{k+1})}$$
$$\beta_{k-1} = \frac{\sum_j \gamma_j \delta_j^{k-1}}{\sum_j \lambda_j \delta_j^{k-2}} \times \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}}$$
$$\vdots$$
$$\beta_\ell = \frac{\sum_j \gamma_j \delta_j^\ell}{\sum_j \gamma_j \delta_j^{\ell-1}} \times \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}}$$
$$\vdots$$
$$\beta_1 = \frac{\sum_j \gamma_j \delta_j}{\sum_j \gamma_j} \times \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}}.$$

Now we need to show that $0 < \beta_\ell < 1$ for each $\ell = 1, \ldots, k$. Let $1 < \ell < k$. Denote by A the term $\left[\sum_j \gamma_j \delta_j^\ell)^2 - (\sum_j \gamma_j \delta_j^{\ell-1})(\sum_j \gamma_j \delta_j^{\ell+1}\right]$. Then, we have

$$\begin{aligned} A =& \sum_j (\gamma_j \delta_j^\ell)^2 + 2\sum_{i<j} \gamma_i \gamma_j \delta_i^\ell \delta_j^\ell - \big(\sum_j (\gamma_j \delta_j)^2 + \sum_{i<j} \gamma_i \gamma_j \delta_i^{\ell-1} \delta_j^{\ell+1} + \sum_{i<j} \gamma_i \gamma_j \delta_i^{\ell+1} \delta_j^{\ell-1}\big) \\ =& \sum_{i<j} \gamma_i \gamma_j (\delta_i \delta_j)^2 \big(2\delta_i \delta_j - \delta_i^2 - \delta_j^2\big) = -\sum_{i<j} \gamma_i \gamma_j (\delta_i \delta_j)^2 \big(\delta_i - \delta_j^2\big)^2 < 0. \end{aligned}$$

Since $A < 0$, this implies

$$\frac{\sum_j \gamma_j \delta_j^\ell}{\sum_j \gamma_j \delta_j^{\ell-1}} \leq \frac{\sum_j \gamma_j \delta_j^{\ell+1}}{\sum_j \gamma_j \delta_j^\ell}.$$

By induction, we obtain

$$\beta_\ell = \frac{\sum_j \gamma_j \delta_j^\ell}{\sum_j \gamma_j \delta_j^{\ell-1}} \times \frac{\sum_j \gamma_j \delta_j^k}{\sum_j \gamma_j \delta_j^{k+1}} \leq \frac{\sum_j \gamma_j \delta_j^{k+1}}{(\sum_j \gamma_j \delta_j)^k} \times \frac{\sum_j \gamma_j \delta_j^k}{(\sum_j \gamma_j \delta_j)^{k+1}} = 1.$$

D.4 Proof of Theorem 4

Note that Theorem 4 is a special case of Theorem 5 in which $k = 3$. Therefore, its proof follows directly from the proof of Theorem 5.

D.5 Proof of Proposition 4

Let $k < k'$. Let α_i and γ_j be nonnegative numbers such that $\sum_{i\in\mathcal{I}} \alpha_i = \sum_{j\in\mathcal{I}} \gamma_j = 1$. Therefore, according to (6,7,8):

$$\begin{aligned} u = \sum_i \alpha_i u_i &= \hat{u} \\ \delta = \frac{\sum_j \gamma_j \delta_j^{k+1}}{\sum_j \gamma_j \delta_j^{k}} &< \hat{\delta} = \frac{\sum_j \gamma_j \delta_j^{\hat{k}+1}}{\sum_j \gamma_j \delta_j^{\hat{k}}} \\ \beta_\ell = \frac{\sum_j \gamma_j \delta_j^{\ell}}{\sum_j \gamma_j \delta_j^{\ell-1}} \times \frac{\sum_j \gamma_j \delta_j^{k}}{\sum_j \gamma_j \delta_j^{k+1}} &> \hat{\beta}_\ell = \frac{\sum_j \gamma_j \delta_j^{\ell}}{\sum_j \gamma_j \delta_j^{\ell-1}} \times \frac{\sum_j \gamma_j \delta_j^{\hat{k}}}{\sum_j \gamma_j \delta_j^{\hat{k}+1}}. \end{aligned}$$

Let $t, s \in \mathbb{N}$ with $t > 1$. Consider

assumptions $x, y, x', y' \in \mathcal{L}$ such that $U(x, \bar{z}_*) = U(y_t, \bar{z}_{*-t})$, $\hat{U}(x', \bar{z}_*) = \hat{U}(y'_t, \bar{z}_{*-t})$ and $U(x_s, \bar{z}_{*-s}) \leq U(y_{t+s}, \bar{z}_{*-(t+s)})$. Equivalently, we have the following:

$$\begin{aligned} u(x) &= d_t u(y) \\ \hat{u}(x') &= \hat{d}_t \hat{u}(y') \\ d_s u(x) &\leq d_{t+s} u(y). \end{aligned}$$

Therefore,

$$d_t d_s \leq d_{t+s}. \tag{11}$$

We must show that $\hat{d}_t \hat{d}_s \leq \hat{d}_{t+s}$. Hence, consider the following three cases:

Case 1: $k \geq t + s$.

Then, (11) implies

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_{t+s}} < \delta.$$

In this case, we know that

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_{t+s}} = \frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{t+s}}.$$

Since $\delta < \hat{\delta}$, it is straightforward that

$$\frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{t+s}} < \hat{\delta},$$

which implies that $\hat{d}_t \hat{d}_s \leq \hat{d}_{t+s}$.

<u>Case 2:</u> $k < t + s \leq \hat{k}$.

Assume that $t \leq k$ and $s \leq k$. (For the case of $t \geq k$ or $s \geq k$, the proof is quite similar.) Then, (11) implies

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_k} < \delta.$$

In this case, we know that

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_k} = \frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{t+s}} \times \left(\frac{\hat{\delta}}{\delta}\right)^{t+s-k} \times \hat{\beta}_{k+1} \cdots \hat{\beta}_{t+s}.$$

Note that for $1 \leq \ell \leq t + s - k$, the following applies:

$$\left(\frac{\hat{\delta}}{\delta}\right) \times \hat{\beta}_{k+\ell} \geq 1,$$

which implies:

$$\left(\frac{\hat{\delta}}{\delta}\right)^{t+s-k} \times \hat{\beta}_{k+1} \cdots \hat{\beta}_{t+s} \geq 1.$$

Therefore:

$$\frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{t+s}} < \hat{\delta}.$$

<u>Case 3:</u> $t + s > \hat{k}$.

We only prove the case corresponding to $t \leq k$ and $s \leq k$ since the rest are similar. Again, we have

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_k} < \delta.$$

In this case, we know that

$$\frac{(\beta_1 \cdots \beta_t)(\beta_1 \cdots \beta_s)}{\beta_1 \cdots \beta_k} = \frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{\hat{k}}} \times \left(\frac{\hat{\delta}}{\delta}\right)^{t+s-\hat{k}} \times \hat{\beta}_{k+1} \cdots \hat{\beta}_{\hat{k}}.$$

According to the same argument as the one used in Case 2, we have

$$\left(\frac{\hat{\delta}}{\delta}\right)^{t+s-k} \times \hat{\beta}_{k+1} \cdots \hat{\beta}_{t+s} \geq 1.$$

Therefore:

$$\frac{(\hat{\beta}_1 \cdots \hat{\beta}_t)(\hat{\beta}_1 \cdots \hat{\beta}_s)}{\hat{\beta}_1 \cdots \hat{\beta}_{\hat{k}}} < \hat{\delta}.$$

D.6 Proof of Corollary 1

The necessity part is immediate. We only show the sufficiency part.

Suppose social lifetime utility U satisfies k-iaiPC for $k > 3$. Since individual lifetime utilities are EDUs, the k-iaiPC is equivalent to k-Unanimity, and $U_{ij}^k = u_i + \delta_j u_i + \cdots + \delta_j^{k-1} u_i$. From Proposition A1 and previous arguments, there exist nonnegative λ_{ij} with $\sum_{ij} \lambda_{ij} = 1$ such that for all $\mathbf{z} \in \mathcal{L}^k$, the following applies:

$$u(z_1) + \delta u(z_2) + \cdots + \delta^{k-1} u(z_k) = \sum_{ij} \lambda_{ij} \Big(u_i(z_1) + \delta_j u_i(z_2) + \cdots + \delta_j^{k-1} u_i(z_k) \Big).$$

Now, let $\alpha_i = \sum_j \lambda_{ij}$ and $\gamma_j = \sum_i \lambda_{ij}$. Then, we obtain

$$u = \sum_i \alpha_i u_i \quad \text{and} \quad \delta^k = \sum_j \gamma_j \delta_j^k \quad \text{for all } k.$$

Therefore, we must have $\gamma_j = 0$ or 1. That is, the rule is dictatorial for the conception of the social discount factor.

REFERENCES

Abi Adams, Laurens Cherchye, Bram De Rock, and Ewout Verriest. Consume Now or Later? Time Inconsistency, Collective Choice, and Revealed Preference. *American Economic Review*, 104(12):4147–83, December 2014.

Arthur E. Attema, Han Bleichrodt, Kirsten I. M. Rohde, and Peter P. Wakker. Time-Tradeoff Sequences for Analyzing Discounting and Time Inconsistency. *Management Science*, 56(11):2015–2030, 2010.

Robert J. Barro. Are Government Bonds Net Wealth? *Journal of Political Economy*, 82(6):1095–1117, November 1974.

B. Douglas Bernheim. Intergenerational Altruism, Dynastic Equilibria and Social Welfare. *The Review of Economic Studies*, 56(1):119–128, 01 1989.

Antoine Billot and Xiangyu Qu. Utilitarian Aggregation with Heterogeneous Beliefs. *American Economic Journal: Microeconomics*, 13(3):112–23, August 2021.

Han Bleichrodt, Kirsten I.M. Rohde, and Peter P. Wakker. Koopmans' Constant Discounting for Intertemporal Choice: A Simplification and a Generalization. *Journal of Mathematical Psychology*, 52(6):341–347, December 2008.

J.M. Buchanan and G. Tullock. *The Calculus of Consent: Logical Foundations of Constitutional Democracy*. Ann Arbor paperbacks. University of Michigan Press, 1962.

Andrew Caplin and John Leahy. The Social Discount Rate. *Journal of Political Economy*, 112(6):1257–1268, 2004.

Christopher P. Chambers and Federico Echenique. On Multiple Discount Rates. *Econometrica*, 86(4):1325–1346, 2018.

Christopher P. Chambers, Federico Echenique, and Alan D. Miller. Decreasing impatience. working paper, 2021.

Graciela Chichilnisky, Peter J. Hammond, and Nicholas Stern. Fundamental utilitarianism and intergenerational equity with extinction discounting. *Social Choice and Welfare*, 54(2):397–427, 2020.

Jonathan Cohen, Keith Marzilli Ericson, David Laibson, and John Myles White. Measuring Time Preferences. *Journal of Economic Literature*, 58(2):299–347, June 2020.

Nicolas Drouhin. Non-stationary additive utility and time consistency. *Journal of Mathematical Economics*, 86:1 – 14, 2020.

Jean-Pierre Drugeon and Bertrand Wigniolle. On Markovian collective choice with heterogeneous quasi-hyperbolic discounting. *Economic Theory*, 2020.

Moritz A. Drupp, Mark C. Freeman, Ben Groom, and Frikk Nesje. Discounting Disentangled. *American Economic Journal: Economic Policy*, 10(4):109–34, November 2018.

Emmanuel Farhi and Iván Werning. Inequality and Social Discounting. *Journal of Political Economy*, 115(3):365–402, 2007.

M. S. Feldstein. The Social Time Preference Discount Rate in Cost Benefit Analysis. *The Economic Journal*, 74(294): 360–379, 1964.

Tangren Feng and Shaowei Ke. Social Discounting and Intergenerational Pareto. *Econometrica*, 86(5):1537–1567, 2018.

Peter C. Fishburn and Ariel Rubinstein. Time Preference. *International Economic Review*, 23(3):677–694, 1982.

Shane Frederick, George Loewenstein, and Ted O'Donoghue. Time Discounting and Time Preference: A Critical Review. *Journal of Economic Literature*, 40(2):351–401, June 2002.

Mark C. Freeman and Ben Groom. Positively Gamma Discounting: Combining the Opinions of Experts on the Social Discount Rate. *The Economic Journal*, 125(585):1015–1024, 07 2014.

Simone Galperti and Bruno Strulovici. A Theory of Intergenerational Altruism. *Econometrica*, 85(4):1175–1218, 2017.

Reyer Gerlagh and Matti Liski. Consistent climate policies. *Journal of the European Economic Association*, 16(1): 1–44, 03 2017.

Itzhak Gilboa, Dov Samet, and David Schmeidler. Utilitarian Aggregation of Beliefs and Tastes. *Journal of Political Economy*, 112(4):932–938, August 2004.

Christian Gollier and Richard Zeckhauser. Aggregation of Heterogeneous Time Preferences. *Journal of Political Economy*, 113(4):878–896, August 2005.

Francisco M. Gonzalez, Itziar Lazkano, and Sjak A. Smulders. Intergenerational altruism with future bias. *Journal of Economic Theory*, 178:436–454, nov 2018.

Yoram Halevy. Time Consistency: Stationarity and Time Invariance: Time Consistency. *Econometrica*, 83(1):335–352, January 2015.

John C Harsanyi. Cardinal utility in welfare economics and in the theory of risk-taking. *Journal of Political Economy*, 61(5):434–435, 1953.

John C. Harsanyi. Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparisons of Utility. *Journal of Political Economy*, 63(4):309–321, 1955.

Bård Harstad. Technology and Time Inconsistency. *Journal of Political Economy*, 128(7):2653–2689, July 2020.

Takashi Hayashi. Quasi-stationary cardinal utility and present bias. *Journal of Economic Theory*, 112(2):343 – 352, 2003.

Takashi Hayashi and Michele Lombardi. Social discount rate: spaces for agreement. *Economic Theory Bulletin*, 08 2021.

Matthew O. Jackson and Leeat Yariv. Present Bias and Collective Dynamic Choice in the Lab. *American Economic Review*, 104(12):4184–4204, December 2014.

Matthew O. Jackson and Leeat Yariv. Collective Dynamic Choice: The Necessity of Time Inconsistency. *American Economic Journal: Microeconomics*, 7(4):150–178, November 2015.

Larry Karp. Global warming and hyperbolic discounting. *Journal of Public Economics*, 89(2):261–282, 2005.

Miles S. Kimball. Making sense of two-sided altruism. *Journal of Monetary Economics*, 20(2):301 – 326, 1987.

Tjalling C. Koopmans. Stationary Ordinal Utility and Impatience. *Econometrica*, 28(2):287, April 1960.

David Laibson. Golden Eggs and Hyperbolic Discounting. *The Quarterly Journal of Economics*, 112(2):443–477, 1997.

David Laibson. Why don't present-biased agents make commitments? *American Economic Review Papers and Proceedings*, 105(5):267–272, 2015.

Stephen A. Marglin. The Social Rate of Discount and The Optimal Rate of Investment. *The Quarterly Journal of Economics*, 77(1):95–111, 1963.

Antony Millner. Nondogmatic Social Discounting. *American Economic Review*, 110(3):760–775, March 2020.

Antony Millner and Geoffrey Heal. Discounting by Committee. *Journal of Public Economics*, 167:91–104, November 2018.

Philippe Mongin. Consistent Bayesian Aggregation. *Journal of Economic Theory*, 66(2):313 – 351, 1995.

J.L. Montiel Olea and Tomasz Strzalecki. Axiomatization and Measurement of Quasi-hyperbolic Discounting. *Quarterly Journal of Economics*, 129:1449–1499, 2014.

Frikk Nesje. Cross-dynastic intergenerational altruism. working paper, 2021.

Jawwad Noor. Hyperbolic discounting and the standard model: Eliciting discount functions. *Journal of Economic Theory*, 144(5):2077 – 2083, 2009.

Jawwad Noor and Norio Takeoka. Optimal discounting. *Econometrica*, 90(2):585–623, March 2022.

William D. Nordhaus. A Review of the Stern Review on the Economics of Climate Change. *Journal of Economic Literature*, 45(3):686–702, September 2007.

E. S. Phelps and R. A. Pollak. On Second-Best National Saving and Game-Equilibrium Growth. *The Review of Economic Studies*, 35(2):185–199, 1968.

Drazen Prelec. Decreasing Impatience: A Criterion for Non-stationary Time Preference and Hyperbolic Discounting. *Scandinavian Journal of Economics*, 106(3):511–532, October 2004.

John K.-H. Quah and Bruno Strulovici. Discounting, Values, and Decisions. *Journal of Political Economy*, 121(5): 896–939, 2013.

F. P. Ramsey. A Mathematical Theory of Saving. *The Economic Journal*, 38(152):543–559, 1928.

Debraj Ray. Hedonistic altruism and welfare economics. working paper, 2018.

Maria Saez-Marti and Jörgen W. Weibull. Discounting and altruism to future decision-makers. *Journal of Economic Theory*, 122(2):254 – 266, 2005.

Paul A. Samuelson. A Note on Measurement of Utility. *The Review of Economic Studies*, 4(2):155–161, 1937.

Richard Thaler. Some empirical evidence on dynamic inconsistency. *Economics Letters*, 8(3):201 – 207, 1981.

Martin L. Weitzman. Gamma Discounting. *American Economic Review*, 91(1):260–271, March 2001.

Stéphane Zuber. The aggregation of preferences: can we ignore the past? *Theory and Decision*, 70(3):367–384, March 2011.