Structured Ambiguity and Sequential Aggregation*

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Abstract

When society notices the possibility for individual priors to be misspecified, it faces a *structured* ambiguity while trying to make a collective decision. Here, we highlight the different principles allowing to overcome such a structured ambiguity by relying on the consensual core of individual priors. This is done through a *sequential* aggregation mechanism. The decision problem at stake is decomposed into several steps and aggregation is made progressively. Compared to standard synchronized aggregation, the possibilities of aggregation are then shown to dramatically increase.

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1 Introduction

Often, society is required to make economic decisions in the presence of ambiguity. Thus, when a social planner has to make certain policy choices, she may lack the

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statistically significant data needed to select the most appropriate course of action. She may also not be able to properly assess the extent to which the models recommended by experts to guide her decisions actually consider all relevant dimensions of the problem she faces. Even more, these models may produce conflicting recommendations. This type of concern is evident, for example, in the formulation of options for environmental economic policy or fiscal policy during a pandemic. With respect to environmental economic policy, it is often difficult to distinguish the underlying mechanisms that govern environmental policy. The clear lack of effective data is a hindrance to building a model that is both credible and consensual.

Experts from different fields have developed a wide variety of climate models, each not only reflecting different design and implementation choices but also making very different predictions of global climate change. How, then, to select models that would correctly calculate the expected loss of each policy and thus identify the optimal policy via a simple cost-benefit analysis. Similarly, at the beginning of the Covid crisis, both theoretical knowledge about the virus and relevant data to help decide on an effective policy were very limited. In a macroeconomic setting, the models used to guide, for example, public debt are again often rather crude simplifications. Hence, this paper focuses on an old theoretical question for social authorities, which concerns how to aggregate the preferences of experts given the ambiguity they face.

To begin, we need to clarify what we mean by *ambiguity* in this paper. According to Hansen (2014)(p. 947), it is possible to distinguish three different situations of uncertainty. A first situation is related to the uncertainty inherent to a particular statistical model. A second concerns uncertainty about the best statistical model to consider in a set of alternative statistical models, and a third situation has to do with the possibility that each element of this set of models is actually misspecified. Here, clearly, we refer to the third form of uncertainty (in the sense of Hansen). Each individual, member of society, has his own statistical model built, among other things, on a probability distribution. Society is then supposed to construct its own *social* model by aggregating these different individual models. The society we consider faces two different sources of uncertainty. The first source of uncertainty relates to the fact that the individual probabilistic beliefs are not consistent with each other, and even vary greatly. This heterogeneity of individual beliefs is widely observed in reality and has given rise to an abundant literature for many years. The second source of uncertainty corresponds to the fact that individuals do not have full confidence in the probability distribution that they use to represent their beliefs. This lack of confidence generally stems from two aspects. First, the process of constructing a model is by nature a simplification and an approximation of the real situation. Therefore, as Hansen says, the possibility of model misspecification can never be completely ruled out. Second, the model validation process suffers from a lack of sufficiently good data, so that the model itself cannot be accurately identified. This is exactly what Manski, Sanstad and DeCanio (2021) calls *partial identification*, as the result of which the related uncertainty is then named *deep uncertainty* or *structured uncertainty*.

We suggest in this paper that society adopts an α-maxmin model (Gilboa and Schmeidler (1989); Ghirardato, Maccheroni and Marinacci (2004); Gul and Pesendorfer (2015)) to deal with the structured uncertainty generated by the aggregation of individual models. We further assume that each individual model corresponds to the *expected utility* (EU) one. Therefore, a society has to build its own model by determining its set of probabilities, selecting its attitude towards ambiguity, *i.e.*, the value α, and its social utility. At the same time, this construction mechanism proceeding by aggregation, it must satisfy certain probing constraints.

First, society considers the set of probability distributions underlying individual beliefs. From this set, events whose individual probabilities are consistent with each other, *i.e.*, are collectively consistent, are identified to form the set of *unambiguous* events. For these events, although society recognizes the possibility that an individual model may be misspecified, it has no really strong argument for deviating from probabilities that are collectively consistent, and so society is assumed to accept them. For ambiguous events, *i.e.*, events for which individual beliefs are inconsistent with each other, society is concerned that the credibility of individual models cannot be properly assessed because it could be that these models are poorly parameterized. In addition, it considers that possibilities other than just the individual probabilities cannot be excluded. Consequently, society is led to exclude no probability that is consistent with that of unambiguous events. In other words, the set of social probabilities consists of all possible extensions of probabilities defined on the unambiguous events. Second, society must specify its attitude towards ambiguity. We do not comment on whether it is better for society to like ambiguity or rather to be resistant to it. In other words, we do not seek to promote certain axioms in order to constrain social attitudes toward ambiguity. While this is an important issue, we believe that more structure, conditions, and information are needed to make a reasonable analysis of this issue, and this is clearly beyond the scope of this paper. Finally, the form of social utility used here is utilitarian. That is, it is defined as the weighted sum of individual utilities. In fact, the aggregation rule illustrated above is to some extent consistent with the proposal of Hansen and Sargent (2022). Indeed, when the individual priors are potentially misspecified, the set of possible priors identified by society should be a superset of the set of these individual priors.

To this end, we adopt a framework similar to Gilboa, Samet and Schmeidler (2004) to describe the ambiguity environment. Clearly, the form of social utility proposed above cannot be justified by applying a traditional aggregation rule and adopting one of the many versions of the Pareto condition that the literature has recently suggested. The new approach to aggregation that we propose can be broken down into several steps. The first step is to apply a Pareto condition *à la* Harsanyi linking individual preferences and social ones. Since the principle only applies to acts defined as lotteries, this ensures that the social probabilities on unambiguous events are collectively consistent while the social utility function corresponds to the weighted average of individual utilities. Because of its lack of confidence in individual beliefs, society completely ignores the perceptions and assessments of individuals about ambiguous events. It rather considers unambiguous events to approximate and estimate ambiguous ones. Although there are many approximation and estimation methods, society is supposed to consider only two of them in order to frame each estimate in an interval. One is an *optimistic* estimate and the other is a *pessimistic* estimate. Hence, for an *ambiguous act*, thanks to the expected utility model, the optimistic estimate equates its expected utility with the lowest expected utility of the unambiguous acts that may dominate it statewisely. Similarly, the pessimistic estimate equates its expected utility with the highest expected utility of the unambiguous acts that are dominated. In this way, society constructs two different models: the first one based on the optimistic estimate is the maxmax model, while the second one based on the pessimistic estimate is the maxmin model. More importantly, both models share the same set of probabilities, *i.e.*, all probability expansions that are based on unambiguous events. Therefore, the next step considers only the optimistic and pessimistic estimates. If society obeys the Pareto principle based on these two models as well as the independence principle, then the social model is exactly the α-maxmin model.

A social utility function defined as a weighted average of individual utilities is likely to define a consensus for most economists. However, for social beliefs that take into account all possible expansions of probability distributions for unambiguous events, it seems that some would then have a different view. In particular, these social beliefs might appear too conservative and less intuitive when individuals do not vary much in their probability estimates of ambiguous events. In such circumstances, it is more reasonable to consider that the domain of social beliefs is actually a subset of the expansions of all probabilities. In this paper, we also consider this case. In fact, two additional steps in the aggregation process are necessary to achieve our goal.

First, through their utility representation, unambiguous social preferences are constructed on the basis of optimistic and pessimistic estimates. It is required that the associated optimistic and pessimistic preferences obey a principle of unambiguous unanimity. It is important to note that the unambiguous social preferences derived here are in fact incomplete. It turns out that preferences can be represented by an EU (Bewley (2002)) in which beliefs are found to belong to some subset of probability expansions. Now, Optimistic and pessimistic preferences can be constructed on the basis of optimistic and pessimistic beliefs. Undoubtedly, aggregating the newly derived preferences, again by applying a principle of unanimity, makes the set of beliefs associated with the α-maximal social preferences a subset of all probability expansions. In this *sequential aggregation* mechanism, society is free to choose any subset of probability expansions: either the set of all individual probabilities, or a single probability, which then reduces the social model to that of standard expected utility.

With the exception of a few rare contributions (Pivato (2022)), theories of aggregation under uncertainty generally rely on methodological individualism. In other words, social values are generally determined by individual values. For example, the social utility function can be a weighted average of the individual functions (Harsanyi (1955)), the social belief can be a weighted average of the individual beliefs (Gilboa, Samet and Schmeidler (2004); Billot and Qu (2021)) or a geometric average (Dietrich (2021)), etc. Eventually, the range of social beliefs is bounded by the set of individual beliefs (Crès, Gilboa and Vieille (2011); Alon and Gayer (2016); Qu (2017); Danan et al. (2016)). Contrary to these previous works, this article requires only a principle of *partial individualism*. That is, it intends to use only those event probabilities for which individuals are collectively consistent, while retaining the other event probabilities incorporating divergent opinions. As discussed earlier, in situations of structured uncertainty, such as environmental issues, individual models often lack sufficient theoretical and empirical evidence to be convincing, and are therefore often subject to misjudgment. The potential for individual models to be misspecified should therefore encourage society to be more careful in choosing which model to use, especially for environmental issues, in order to avoid future catastrophes. Therefore, the main strength of this paper is to provide a novel and reasonable method for a society to aggregate individual preferences when the models employed by individuals are likely to be misspecified.

The other major innovation of this article is the sequential aggregation mechanism. Indeed, it does not seem that there are any other paper using such a mechanism. It is well known that synchronized aggregation, *i.e.*, a *one-shot* mechanism, usually creates difficulties affecting the aggregation result. For example, Mongin (1995) and Mongin and Pivato (2020) highlight that unanimity in synchronized aggregation may be *spurious*. In the case of sequential aggregation, however, it is possible to decompose the problem into several steps and aggregate the parameters we need in succession. Here, we use the sequential aggregation principle, first for the social utility function, then for the social beliefs, and thus obtain different aggregated beliefs per iteration. Compared to synchronized aggregation, sequential aggregation efficiently extends the possible outcomes of aggregation. In reality, social decisions are not made piecemeal, but are rather the result of repeated refinement. In this sense, sequential aggregation is a more accurate description of real process of social decisions.

This paper is organized as follows. Section 2 contains the framework and aggregation result for the social utility function. Section 3 formally outlines and investigates the sequential aggregation method and presents the main result. Section 4 considers some extensions of this result. We conclude in Section 5. All proofs are in the Appendix.

2 The Model

Let (S, Σ) be a σ-measurable space, where S, a set of *states*, is a separable metric space and Σ is a σ-algebra of *events*. Denote by X a set of *outcomes*, which is assumed to be a connected and compact metric. A typical social *act* is a Σ-measurable *simple* function $f : S \to X$ and $\mathcal{F}$ is the set of all social acts.[1] Society is a set of individuals $\mathcal{I} = \{1, \ldots, n\}$. Each individual $i \in \mathcal{I}$ has preferences over $\mathcal{F} \times \mathcal{F}$, that is a binary relation $\succsim_i \subset \mathcal{F} \times \mathcal{F}$. Social preferences are denoted by $\succsim \subset \mathcal{F} \times \mathcal{F}$. A function $V : \mathcal{F} \to \mathbb{R}$ represents preferences $\succsim$ on $\mathcal{F}$ if, for all $f, g \in \mathcal{F}$, $f \succsim g$ if and only if $V(f) \geq V(g)$.

Definition 1. A function $V : \mathcal{F} \to \mathbb{R}$ is a *subjective expected utility* (SEU) function if there exists a unique countably additive, non-atomic probability measure π on Σ, and a continuous utility function u on X, s.t., for $f \in \mathcal{F}$:

$$V(f) = \int_S u(f) \mathrm{d}\pi. \tag{1}$$

Assumption 1 — SEU individuals. Each individual preferences $\succsim_i$ admits a Savage Expected Utility (SEU) representation, *i.e.*, (u_i, π_i) are the unique pair that represents $\succsim_i$ as in (1).

Assumption 1 requires that all individual preferences satisfy Savage's postulates and, therefore, are all represented by a SEU. In contrast, in the presence of heterogeneous individual beliefs, we do not impose that social preferences admit a SEU

[1] The *topology of pointwise convergence* on $\mathcal{F}$ is defined as the relative topology with respect to the product topology on X^S.

representation. This is the main difference between our model and Gilboa, Samet and Schmeidler (2004) and Mongin (1995). Of course, under objective uncertainty, Harsanyi (1955) assumes that social preferences are represented by an expected utility function. When all individuals agree on all events, there is no conflict between beliefs. It is then natural to suppose that both individuals and society admit the same form of preference representation. However, when there is a conflict between individual beliefs, society has to compromise with these divergent beliefs. It is neither intuitive nor reasonable to determine the form of representation before setting the rule for aggregating beliefs (See, for instance, Diamond (1967)). For this reason, it seems that it is more appropriate to assume that social preferences admit a SEU representation that is limited to those acts for which all individuals agree on the corresponding events while, for other acts, it remained 'representation agnostic'.

Definition 2. An event A is *unambiguous* if $\pi_i(A) = \pi_j(A)$, for all $i, j \in \mathcal{I}$.

Let $\mathscr{A}$ be the set of all unambiguous events.[2] Thus, an event A is in $\mathscr{A}$ if all individuals agree on its probability. An act f is a *lottery* if each measurable subset of outcomes Y is unambiguous, *i.e.*, $f^{-1}(Y) \in \mathscr{A}$, for all $Y \subset X$. We denote by $\mathcal{L}$ the set of lotteries. All individuals agree on the probability of the events used to define a lottery, and if all individuals agree on all events, then $\mathscr{A} = \Sigma$. However, if there is heterogeneity of beliefs, then the set of unambiguous events is only a subset of Σ and, in general, does not define an algebra. Let us introduce a λ-*system* $\Lambda \subseteq 2^{\Omega}$ as a collection of subsets such that:

(i) $\Omega \in \Lambda$,

(ii) if $E \in \Lambda$, then $E^c \in \Lambda$, and

(iii) for any countable collection of disjoint events $E_k \in \Lambda$, $\bigcup_k E_k \in \Lambda$.

Lemma 1. *The collection $\mathscr{A}$ of unambiguous events is a λ-system.*

(All proofs are in the Appendix.)

We say that $\pi : \mathscr{A} \to [0, 1]$ is a probability measure on unambiguous events if:

(i) $\pi(\emptyset) = 0 \leq \pi(A) \leq \pi(S) = 1$, for every $A \in \mathscr{A}$, and

[2] An unambiguous event is also an 'unanimous' event in terms of individual beliefs.

(ii) $\pi(\bigcup_k A_k) = \sum_k \pi(A_k)$, for any countable collection of disjoint events $A_k \in \mathscr{A}$.

Note that any probability measure on Σ restricted to $\mathscr{A}$ is a probability measure on $\mathscr{A}$. Thus, π, that is the probability measure on unambiguous events defined by $\pi(A) = \pi_i(A)$, for all $A \in \mathscr{A}$, is a probability measure on $\mathscr{A}$.

Definition 3. Given a collection of unambiguous events $\mathscr{A}$, a function $V : \mathcal{F} \to \mathbb{R}$ is a *restricted* SEU (RSEU) with respect to $\mathscr{A}$ if there exist a probability measure π on $\mathscr{A}$ and a continuous utility function u on X, such that, for $f \in \mathcal{L}$:

$$V(f) = \int_S u(f)\mathrm{d}\pi,$$

and if it is monotonic[3] and continuous (in the topology of pointwise convergence).

Assumption 2 — RSEU society. Social preferences $\succsim$ admit a RSEU representation, *i.e.*, $(\mathscr{A}, \pi, u)$ is the unique triplet that represents $\succsim$ (up to the affine transformation of u).

By convenience, we say in an undifferentiated way that V, the function that represents $\succsim$, is a RSEU or that the triplet $(\mathscr{A}, \pi, u)$ is a RSEU. Accordingly, Assumption 2 requires that social preferences $\succsim$ satisfy Savage's postulates restricted to the lottery set.[4]

Society is concerned with resolving disagreements between individuals and, therefore, can only really commit to Bayesian behavior for lotteries.

Harsanyi Pareto condition (HPC). For every lotteries $f, g \in \mathcal{L}$, if $f \succsim_i g$, for all i, then $f \succsim g$.

HPC requires that if all individuals prefer a first lottery to a second lottery, then society also prefers the first lottery. In terms of individual beliefs, each lottery corresponds to an identical von Neumann-Morgenstern (vNM) lottery. Therefore, HPC

[3] V is *monotonic* if $u(f) \geq u(g)$ implies $V(f) \geq V(g)$.

[4] We refer to Epstein and Zhang (2001) and Kopylov (2007) for the formal characterization of a RSEU.

can be interpreted as a natural extension of the Harsanyi-like objective uncertainty condition to the subjective uncertainty model.

Definition 4. Given $\{(u_i, \pi_i)\}_{i \in \mathcal{I}}$, a RSEU $\{(\mathscr{A}, u, \pi)\}$ is *collectively consistent* if $\mathscr{A}$ is the set of unambiguous events with $\pi(A) = \pi_i(A)$, for all i and all $A \in \mathscr{A}$, and it is *utilitarian* if u is a convex combination of $\{u_i\}_{i \in \mathcal{I}}$. Moreover, a RSEU is said to be *consistently utilitarian* if it is both collectively consistent and utilitarian.

The next result characterizes the relation between HPC and a consistently utilitarian RSEU.

Theorem 1. HPC *holds if and only if social preferences $\succsim$ are represented by a consistently utilitarian RSEU.*

This theorem does not suggest a specific form of representation for all acts in general. It only restricts social utility to be a convex combination of individual utilities and social beliefs based on unambiguous events to be consistent with individual beliefs. Since we consider here only the representation of social expected utility restricted to lotteries, this result can be seen as a minimal extension of Harsanyi's result from objective uncertainty to subjective uncertainty.

3 Doctrines and Social Opinions

Since all individuals share the same beliefs about each lottery, social preferences about lotteries, based on HPC, can be conceived as a consensus evaluation of lotteries. Note that society agrees only on unambiguous events and thus makes no judgment about events with heterogeneous estimates.[5] If there is no utilitarian controversy about individuals' tastes, then restricted social preferences should be consensually agreed upon by all individuals. Consequently, these preferences could serve as a reference for society when it seeks to estimate acts that are not related to a lottery.

Since the social classification of lotteries is based on an evaluation of beliefs that is common to all individuals, any group or party in a given society can thus

[5] By convenience, these events can sometimes be said *socially ambiguous* or simply *ambiguous*.

distinguish itself by defending its own system of evaluation of any option. This system, specific to a particular group, constitutes a *doctrine*. A doctrine can be, for instance, naively cautious when its system of evaluation is such that any option is considered as socially indifferent to the worst outcome that it implies. But a doctrine can of course be much more sophisticated. Finally, any doctrine is socially legitimate as long as it leads to a consistently utilitarian RSEU preferences.

Formally, social preferences $\succsim$ are said to *match* with preferences $\succsim^*$ on $\mathcal{F}$ if $\succsim$ agrees with $\succsim^*$ on lotteries, *i.e.*, $f \succsim g$ if and only if $f \succsim^* g$, for all $f, g \in \mathcal{L}$. Thus, if social preferences $\succsim$ match with $\succsim^*$, then $(\mathscr{A}, \pi, u)$ represents $\succsim^*$ restricted to $\mathcal{L}$. We say preferences are defining a *social opinion* if they match with $\succsim$ and if, as they are issued from a doctrine, they admit a RSEU representation. Let Θ be a collection of *considerate* social opinions. For any opinion $\theta \in \Theta$, $\succsim_\theta$ denotes the social opinion to which the θ opinion corresponds. The social opinions have an identical ranking of the lotteries. However, they differ not only in their estimates of events that are socially ambiguous, but also in their attitudes toward these events.

For instance, a consistent SEU is a social opinion. Given $(\mathscr{A}, u, \pi)$, let $\mathbb{P}_\pi$ be the set of all the *extensions* of π on Σ. We say a social opinion θ is *probabilistic* if there is $p \in \mathbb{P}_\pi$ such that:

$$f \succsim_\theta g \Leftrightarrow \int u(f)\mathrm{d}p \geq \int u(g)\mathrm{d}p.$$

In this case, we can write $\theta = p$. In a similar manner, a social opinion can be *multiple prior*-EU (MEU) in the sense of Gilboa and Schmeidler (1989).

Because of the different doctrines that are expressed, society faces several opinions that are both rational and mutually contradictory. We argue here that society should respect the following two principles if it intends to formalize its preferences.

Unanimity. For all acts $f, g \in \mathcal{F}$, if $f \succsim_\theta g$, for all $\theta \in \Theta$, then $f \succsim g$.

Unanimity states that if, for all social opinions, f is preferred to g, then society also prefers f. It is a more binding principle for a society than it seems, since no spurious unanimity in the sense of Mongin (1995) can indeed arise thanks to the consensus of social opinions on the ranking of lotteries.

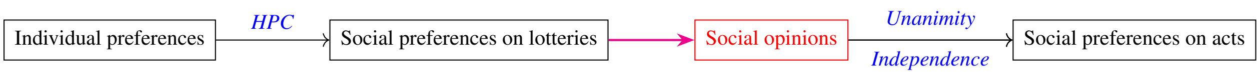

Figure 1: Preference Aggregation Process

For $x, y \in X$ and event A, a binary act, written xAy, describes an act such that the outcome is x if event A is realized and y otherwise.

Independence. For all acts $f, g \in \mathcal{F}$, $x, y \in X$ and $A \in \mathscr{A}$, if for every $\theta \in \Theta$, there exists $z \in X$ such that $f \sim_\theta z$ and $g \sim_\theta zAy$, then $f \sim x$ implies $g \sim xAy$.

Independence states that if, for all social opinions, z is a certainty equivalent of f and the act g is indifferent to the binary act zAy, then that society is indifferent between f and x implies that society is also indifferent between g and xAy. Since A is an unambiguous event, each social opinion $\succsim_\theta$ measures the difference between f and g by means of $\big(1-\pi(A)\big)$ and y and, in addition, society measures the difference between these two acts from the same elements. Note that this principle applies only to unambiguous events. Indeed, zAy is a lottery only if A is an unambiguous event. Social opinions being SEU towards lotteries, therefore, they evaluate the act g separately from z and y.

Our proposed aggregation process is summarized above in Figure 1.

Remark. In the case of probabilistic social opinions, *i.e.*, when $\Theta = \mathbb{P}_\pi$, it becomes necessary to apply a probabilistic principle of unanimity of the following form: "for any $f, g \in \mathcal{F}$, if $f \succsim_\theta g$, for any $\theta \in \mathbb{P}_\pi$, then $f \succsim g$". However, we can immediately see that this puts no restrictions on social preferences, which limits the interest of the process we propose. Therefore, it is natural to consider a subset of all the probabilistic social opinions.

3.1 Pole Opinions

In an environment where societies are constantly inundated with mostly opinionated information via social networks, it is increasingly difficult for individuals to make independent decisions. It is therefore both prudent and effective to focus on the most polar opinions, those that are based on an extreme apprehension of the environment, which therefore consider in a privileged way the best and worst possible scenarios. These opinions are thus supposed to reflect two doctrines, one called *conservative*, the other *progressive*.

Definition 5. A social opinion is *conservative*, written $\succsim_{cons}$ if, for any act $f \in \mathcal{F}$ and $x \in X$, $f \succ_{cons} x$ whence there is a lottery $g \in \mathcal{L}$ such that $u(f) \geq u(g)$ and $g \succ_{cons} x$.

Conservative social opinion is characterized by a cautious way of valuing each option f. Here, the expected utility of an option f is achieved by a sort of approximation-from-below-process such that it is equal to the *highest* expected utility associated to the f-dominated lotteries. As an illustration, let us consider the case of a comparison between an option f and a constant act x. The conservative social opinion does not allow a direct comparison of f with x. Instead, it proposes to make only an indirect comparison through lotteries that are dominated by f. Since monotonicity immediately implies that f is preferred to all lotteries that it dominates, it follows that, if any dominated lottery is preferred to x, then, by transitivity, the option f is also preferred to x.

Definition 6. A social opinion $\succsim_{prog}$ is *progressive* if, for any act $f \in \mathcal{F}$ and $x \in X$, $x \succ_{prog} f$ whence there is a lottery $g \in \mathcal{L}$ such that $u(g) \geq u(f)$ and $x \succ_{prog} g$.

Similarly, a progressive social opinion cannot directly compare option f with constant act x. It also applies an indirect comparison. However, a progressive opinion uses the dominant lotteries to evaluate f in a reckless way this time. If x is preferred to any dominant lottery, then x is also preferred to option f. According to this view, the expected utility of an option is equal, through an approximation-from-above-process, to the *lowest* expected utility associated to the dominant lotteries.

Why should a society give special consideration to conservative and progressive opinions? First, both opinions respect the monotonicity and transitivity of preferences, which ensures that they are indeed rational in the theoretical sense. Second, these two opinions together determine the lower and upper bounds between which the utility of an option can vary. The conservative opinion determines the lower value and the progressive opinion the upper value. Taking these two opinions into account thus avoids over- or under-estimating the utility of an option. Finally, and most importantly, the difference between the two values of an option's expected utility could serve as a natural measure of conflict of opinion. Therefore, it can be expected that society will seek to manipulate this gap in order to strategically use the extent of the conflict of opinion.

Let $\mathbb{P}_\pi$ be the collection of all probability extensions of a probability π. It is easy to see that, for the conservative opinion, if $f \succ_{cons} x$, then the expected utility of f is greater than the expected utility of x, for all extensions of π. Therefore, we can also interpret the conservative opinion such that an option f is preferred to a constant act x only if its expected utility is greater than the expected utility of x, for every possible extension of π. Similarly, for the progressive opinion, if $x \succ_{prog} f$, then the expected utility of f is smaller than the expected utility of x, for every possible extension of π.

Given a subset $\mathbb{P} \subseteq \mathbb{P}_\pi$ of probability extensions of π, we can now define the notion of generalized maxmin expected utility.

Definition 7. A function $V : \mathcal{F} \to \mathbb{R}$ is a *generalized maxmin expected utility* (GMEU) if there exist a nonempty compact and convex set $\mathbb{P}$ of probabilities on Σ, a utility function u on X and a monotonic function $W : u(X) \times u(X) \to \mathbb{R}$ with $V(x) = W\big(u(x), u(x)\big)$, for all $x \in X$, such that:

$$V(f) = W(\overline{u}^f_{\mathbb{P}}, \underline{u}^f_{\mathbb{P}}), \tag{2}$$

where

$$\overline{u}^f_{\mathbb{P}} \equiv \max_{p\in\mathbb{P}} \int u(f)\mathrm{d}p \quad \text{and} \quad \underline{u}^f_{\mathbb{P}} \equiv \min_{p\in\mathbb{P}} \int u(f)\mathrm{d}p.$$

Remark. Given $(\mathscr{A}, u, \pi)$, a GMEU function V is a *generalized Hurwicz expected utility* (GHEU) if $\mathbb{P} = \mathbb{P}_\pi$.

Remark. The definition of a GMEU depends on *polar* expected utilities, in the sense that the utility of an act f is determined only by its highest possible expected utility $\overline{u}^f_{\mathbb{P}}$ and its lowest possible expected utility $\underline{u}^f_{\mathbb{P}}$. A particular occurrence of GMEU is the case of α-MEU utilities:

Definition 8. A function $V : \mathcal{F} \to \mathbb{R}$ is a *α-maxmin expected utility* (α-MEU) if there exist a nonempty compact and convex set $\mathbb{P}$ of probabilities on Σ and a utility function u on X such that:

$$V(f) = \alpha \max_{p \in \mathbb{P}} \int u(f) \mathrm{d}p + (1-\alpha) \min_{p \in \mathbb{P}} \int u(f) \mathrm{d}p, \tag{3}$$

where $\alpha \in [0, 1]$.

Remark. Given $(\mathscr{A}, u, \pi)$, an α-MEU function V is a *Hurwicz expected utility* (HEU) if $\mathbb{P} = \mathbb{P}_\pi$.

Remark. Note that α-MEU includes standard MEU and maxmaxEU as special cases where $\alpha = 0$ and $\alpha = 1$, respectively.

The following theorem shows that if society is only sensitive to conservative and progressive opinions, then the satisfaction of Unanimity is equivalent to the existence of uniformly utilitarian GHEU social preferences.

Theorem 2. *Suppose* $\Theta = \{cons, prog\}$. Unanimity *holds if and only if social preferences* $\succsim$ *are represented by a consistently utilitarian GHEU.*

A special case of Theorem 2 concerns a society that is sensitive to only one opinion. For example, if social sensitivity were conservative, then social preferences would be represented by a standard MEU.

Corollary 1. *Social preferences are conservative,* i.e., $\succsim = \succsim_{cons}$, *iff they are represented by a consistently utilitarian MEU with* $\mathbb{P} = \mathbb{P}_\pi$. *In the same way, social preferences are progressive,* i.e., $\succsim = \succsim_{prog}$, *iff they are represented by a consistently utilitarian maxmaxEU with* $\mathbb{P} = \mathbb{P}_\pi$.

Corollary 1 shows that conservative and progressive opinions are based on the same basic beliefs, which correspond to $\mathbb{P}_\pi$. In reality, they differ only in their system of evaluating non-lottery acts. Unanimity requires that society considers only two opinions when evaluating actions, but it does not specify a precise functional form for expected utility. This, on the one hand, gives society some flexibility in choosing the function to represent its preferences in a complex environment. This, on the other hand, allows society to remain undecided about the way it intends to make its decisions. To derive a more concrete form of representation, such as a α-MEU, Independence is then necessary.

Thus, Theorem 3 below provides an axiomatic characterization of social preferences of type HEU. When society respects both Unanimity and Independence, then social preferences admit a representation of type α-MEU, where social utility is defined as a convex combination of individual utilities and social beliefs as any extension of the probability π.

Theorem 3. *Suppose* $\Theta = \{cons, prog\}$. Unanimity *and* Independence *hold if and only if social preferences* $\succsim$ *are represented by a consistently utilitarian HEU.*

This result characterizes a society that constructs its preferences on the basis of a weighted average of conservative and progressive opinions. The set of socially ambiguous events coincide with those shared by the two opinions. Thus, society advocates that the occurrence of an ambiguous event A should not be evaluated by a single number, but rather by an interval of numbers. Roughly speaking, the lower and upper bounds of this interval are then measured by the probability of the largest unambiguous event contained in A and the smallest unambiguous event containing A, respectively. One can thus say that the social approach to understanding 'conflicting' events, or socially ambiguous events, is, in a way, identical to the approach proposed by Dempster-Shafer for understanding individual uncertainty.

Judging by the widely observed phenomenon of 'ambiguity aversion', it is questionable whether progressive opinion is relevant to the construction of social preferences. It is quite conceivable that conservative opinion is more frequent and progressive opinion is indeed more rare. However, in a collective decision-making framework, as Will Durant pointed out, it is often the case that "a united minority acting against a divided majority" can be influential. Therefore, it is legitimate to think that a HEU society is descriptively more plausible than a society that would a priori have been described as MEU.

3.2 Act Mixture and Opinion Refinement

The conservative and progressive opinions can be considered too extreme. For an ambiguous event, its conservative estimate is actually even lower than the most cautious individual estimate. Similarly, its progressive estimate is greater than the most reckless of the individual estimates. In this subsection, we seek to identify a possible way to refine the conservative and progressive opinions to avoid an overly radical social estimate of these ambiguous events.

We know that the conservative and progressive opinions draw from the same set of social beliefs, *i.e.*, the set of all probability extensions of π. Therefore, the two opinions are likely to share the same *unambiguous preference relation*, which we define below.

Definition 9. For $f, g, h \in \mathcal{F}$ and $A \in \mathscr{A}$, h is a A-*mixture* of f, g if, for all s, $h(s) \sim f(s)Ag(s)$.

We denote the A-mixture of f and g by $f[A]g$.

Remark. For any preferences $\succsim^*$ on $\mathcal{F}$ matching with $\succsim$, the A-mixture of f and g with respect to $\succsim^*$ is the same as the one with respect to $\succsim$.

Definition 10. Given a social opinion $\theta \in \Theta$ and two acts $f, g \in \mathcal{F}$, f is said to be *unambiguously preferred* to g, denoted $f \trianglerighteq_\theta g$, iff, for any $h \in \mathcal{F}$:

$$f \trianglerighteq_\theta g \Longleftrightarrow f[A]h \succsim_\theta g[A]h, \text{ for all } A \in \mathscr{A}.$$

An act f is unambiguously preferred to another act g if all event-mixtures of f and g with another act h reproduce between them the same ranking. We note that the social opinion $\succsim_\theta$, for $\theta \in \{cons, prog\}$, restricted to lotteries is an unambiguous preference. However, in general, $\trianglerighteq_\theta$ is defined on $\mathcal{F}$ and not intended to be restricted to lotteries only. For example, if one option state-wisely dominates another option, it is obvious that the former will be unambiguously preferred to the latter. Similarly, society prefers unambiguously f to g, written $f \trianglerighteq g$, if $f[A]h \succsim g[A]h$, for all $h \in \mathcal{F}$ and $A \in \mathscr{A}$. Since conservative and progressive opinions share the same beliefs, it is imperative to require that if they both believe that one act is unambiguously preferable to another, so does society:

Unambiguous unanimity. If $f \trianglerighteq_\theta g$, for all $\theta \in \{cons, prog\}$, then $f \trianglerighteq g$.

What does unambiguous unanimity imply? First, unambiguous preferences are a priori incomplete. Therefore, social unambiguous preferences $\trianglerighteq$ are Bewley preferences: there exists a subset $\mathbb{P}$ of $\mathbb{P}_\pi$ such that one act is unambiguously preferred to another if and only if the expected utility of the former is greater than that of the latter, this for each probability in $\mathbb{P}$. If society respects Unambiguous unanimity, then unambiguous social preferences can be interpreted as a reference point for the refinement of social opinions.

Definition 11. A social opinion is *pessimistic*, *i.e.*, $\theta = p$, if $f \ntrianglerighteq x$ implies $x \succsim_p f$.

A pessimistic opinion holds that a constant act x is always better than any act f whenever f is not unambiguously preferred to x. In other words, when there is an event A such that an $A-$mixture with x is better than the same mixture with f, then the pessimistic opinion believes that x is a better act.

Definition 12. A social opinion is *optimistic*, *i.e.*, $\theta = o$, if $x \ntrianglerighteq f$ implies $f \succsim_o x$

An optimistic opinion believes that any act f is better than a constant act x if x is not unambiguously preferred to f. In other words, whenever there is an event whose associated mixture with f is better than the same mixture with x, then the optimistic opinion believes that f is a better act.

See Figure 2 above for the formation of pessimistic and optimistic opinions.

We want to emphasize here that both pessimistic and optimistic opinions depend on unambiguous social preferences $\trianglerighteq$. Unambiguous unanimity requires only that $\trianglerighteq$ be Bewley-type preferences characterized by $\mathbb{P} \subseteq \mathbb{P}_\pi$. Therefore, $\mathbb{P}$ may as well contain all probability extensions of π or be a singleton. If $\mathbb{P} = \mathbb{P}_\pi$, then the pessimistic and optimistic opinions are absolutely identical to the conservative and progressive opinions, respectively. If $\mathbb{P}$ is a singleton, then the pessimistic and optimistic opinions coincide and constitute an opinion that can be represented by a SEU, *i.e.*, what we will call later a *Bayesian* opinion.

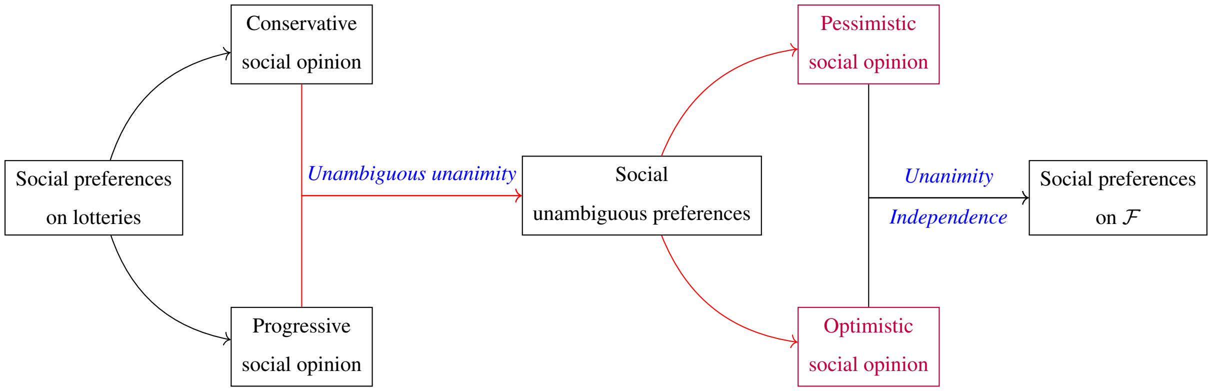

Figure 2: Opinion Process

In this process, society takes the initiative to influence social opinions. If the heterogeneity of individual estimates of an ambiguous event is low enough, society may agree to treat it as an unambiguous event by assigning it a number. As a result, social opinions will not conflict with respect to the estimation of this event. Similarly, society can use the minimum and maximum individual probabilities to estimate each ambiguous event, which will then lead to define $\mathbb{P}$ as the convex hull of the individual probabilities.

The following theorem proposes a formal characterization of a GMEU, which is a generalization of a GHEU. If society considers only pessimistic and optimistic opinions, then Unanimity implies that society admits for its preferences a consistently utilitarian GMEU representation.

Theorem 4. *Suppose* $\Theta = \{p, o\}$. Unanimity *holds if and only if social preferences* $\succsim$ *are represented by a consistently utilitarian GMEU with* $\mathbb{P} \subseteq \mathbb{P}_\pi$.

Theorem 5 states that if, additionally, Independence is satisfied, then society must have a consistently utilitarian α-MEU representation.

Theorem 5. *Suppose* $\Theta = \{p, o\}$. Unanimity *and* Independence *hold if and only if social preferences* $\succsim$ *are represented by a consistently utilitarian* α*-MEU with* $\mathbb{P} \subseteq \mathbb{P}_\pi$.

Theorem 5 is a general result, which includes many important results as special cases. If $\mathbb{P}$ is a singleton all being a convex combination of individual beliefs with

$\alpha = 0$, then it coincides with the theorem in Gilboa, Samet and Schmeidler (2004). Similarly, if $\mathbb{P}$ is a set consisting of all convex combinations of individual beliefs with $\alpha = 0$, then it coincides with the result of Alon and Gayer (2016).

4 Social Opinions and Contamination

In the previous sections, we only considered social opinions that were based on lotteries or unambiguous preferences. However, in some situations, individual beliefs or a weighted average of individual beliefs define undoubtedly a reasonable social opinion. For example, society may be particularly interested in the opinion of a renowned expert. In this case, society may wish to include this expert opinion in the set of possible social opinions.

Definition 13. A social opinion is *Bayesian*, *i.e.*, $\theta = B$, if $\succsim_B$ admits a SEU representation and satisfies HPC.

To be a social opinion, Bayesian preferences $\succsim_B$ must first match with social preferences $\succsim$ regarding lotteries. Moreover, the Bayesian opinion is also associated to a probability measure π_B on Σ, the algebra of possible events. Since $\succsim_B$ satisfies HPC, we can apply the theorem of Gilboa, Samet and Schmeidler (2004), which immediately implies that Bayesian beliefs π_B are defined as a convex combination of individual beliefs.

Proposition 1 (Gilboa, Samet and Schmeidler (2004) Theorem)**.** *If a social opinion is Bayesian, then π_B on Σ is a convex combination of individual beliefs $\{\pi_i\}_{i=1}^n$.*

Now we seek to consider social opinions more holistically instead of separately. According to the above principles, two sharp opinions, such as conservative and progressive, can together form a moderate social opinion. As Theorem 3 shows, a moderate opinion can be represented by a consistently utilitarian HEU.

Definition 14. A social opinion is *moderate*, $\theta = m$, if $\succsim_m$ admits a consistently utilitarian HEU representation.

Let's introduce now the notion of *contamination* when applied to a MEU.

Definition 15. Given $(\mathscr{A}, u, \pi)$, a function $V : \mathcal{F} \to \mathbb{R}$ is a *contamination* maxmin expected utility (cMEU) if there exist a probability measure p on Σ, a nonempty compact and convex set $\mathbb{P}$ of probabilities on Σ and a utility function u on X such that:

$$V(f) = \epsilon \int u(f)\mathrm{d}p + \gamma \max_{q \in \mathbb{P}} \int u(f)\mathrm{d}q + (1 - \epsilon - \gamma) \min_{q \in \mathbb{P}} \int u(f)\mathrm{d}q, \tag{4}$$

where $\epsilon, \gamma \in [0, 1]$ and $\epsilon + \gamma \leq 1$.

Theorem 6 characterizes cMEU social preferences by satisfying Unanimity and Independence applied to both Bayesian and moderate social opinions.

Theorem 6. *Suppose* $\Theta = \{B, m\}$. Unanimity *and* Independence *hold if and only if social preferences* $\succsim$ *are represented by a consistently utilitarian cMEU with* $\mathbb{P} = \mathbb{P}_\pi$ *and* p *is a convex combination of* $\{\pi_i\}_{i=1}^n$.

Similarly, we could consider another aggregate opinion, the so-called *neutral* social opinion, which would aggregate pessimistic and optimistic opinions in the same way that the Bayesian opinion aggregates conservative and progressive opinions. Then, by Unanimity and Independence with respect to Bayesian and neutral opinions, we could characterize a consistently utilitarian cMEU by a smaller set of social beliefs, *i.e.*, $\mathbb{P} \subset \mathbb{P}_\pi$.

Note that our analysis at this point focuses only on a binary opposition of social opinions. What would happen if we expanded the set of social opinions to include all of the opinions we discussed above other than in pairs of antagonistic opinions? As observed by Crès, Gilboa and Vieille (2011) and Qu (2017), in the case of multiple opinions, a stronger axiom than Unanimity and Independence is then needed to characterize the representation.

In Theorem 6, we implicitly assume that opinions aggregate sequentially. We first aggregate conservative and progressive opinions into a moderate opinion. Next, we aggregate moderate and Bayesian opinions into cMEU social preferences. However, the order of aggregation of opinions does not affect the representation of social preferences. At each step of this sequential aggregation, the binary opposition of the opinions leads to build a new opinion referring to a kind of linear average of the two antagonistic opinions. Therefore, as long as the aggregated opinions proceed from a binary opposition, the additive form is maintained

5 Conclusion

Analyzing the conditions for collective economic decision-making is a difficult problem. Economists have developed many models that should logically allow decision-makers, after studying the nature of the practical problems they encounter, to appropriate the right model and, consequently, to make more efficient social decisions. However, decision-makers are often confronted with a double complexity: that of the scientific systems they are asked to study, as is the case for the analysis of global warming for example, and that of the economic models with which they are asked to build an efficient representation of their preferences and the choices available to them. This double complexity generates a considerable structured uncertainty. The major consequence of structured uncertainty is that, even if many different models have been developed, no one can ever be totally free of the risk of misspecification...

It is therefore crucial that this kind of structured uncertainty be taken into account at the very heart of the social decision-making mechanism, and in the study of political choices. The drawbacks of the most famous methods currently used, such as linear aggregation, are well known and particularly well documented in the literature. We conceive our contribution as an alternative approach. We propose an aggregation process that is based on a principle of partial individualism that is less restrictive than standard methodological individualism. Our results suggest that a model of social belief formation based on the consideration of unambiguous and unanimous individual beliefs is a promising way to develop and address how society can make decisions when it is known that individual models are generally misspecified.

A APPENDIX — PRELIMINARIES AND PROOF OF LEMMA 1

Let $B_0(\Sigma)$ is the vector space generated by the indicator functions of the elements of Σ, endowed with the supnorm. We denote by $ba(\Sigma)$ the set of all bounded, finitely additive set functions on Σ, and by $\Delta(\Sigma)$ the set of all probabilities on Σ. We know that $ba(\Sigma)$, endowed with the total variation norm, is isometrically isomorphic to the norm dual of $B_0(\Sigma)$. Therefore, the weak topology, w^*, of $ba(\Sigma)$ coincides with the eventwise convergence topology. Given a nonsingleton interval K in the real line, $B_0(\Sigma, K)$ is the set of the functions in $B_0(\Sigma)$ taking values in K.

We recall that a binary relation $\succsim$ on $B_0(\Sigma, K)$ is:

- *preordered* if it is reflexive and transitive;
- *continuous* if $\varphi_n \succsim \phi_n$, for all $n \in \mathbb{N}$, $\varphi_n \to \varphi$ and $\phi_n \to \phi$ imply $\varphi \succsim \phi$;
- *Archimedean* if the sets $\{\lambda \in [0,1] : \lambda\varphi + (1-\lambda)\phi \succeq \eta\}$ and $\{\lambda \in [0,1] : \eta \succsim \lambda\varphi + (1-\lambda)\phi\}$ are closed in $[0,1]$, for all $\varphi, \phi, \eta \in B_0(\Sigma, K)$;
- *affine* if, for all $\varphi, \phi, \eta \in B_0(\Sigma, K)$ and all $\alpha \in (0,1)$, $\varphi \succsim \phi$ iff $\alpha\varphi + (1-\alpha)\eta \succsim \alpha\phi + (1-\alpha)\eta$;
- *monotonic* if $\varphi \geq \phi$ implies $\varphi \succsim \phi$;
- *nontrivial* if there exists $\varphi, \phi, \eta \in B_0(\Sigma, K)$ such that $\varphi \succsim \phi$ but not $\phi \succsim \varphi$.

Lemma A1. *A binary relation $\succsim$ is a nontrivial, continuous, affine, and monotonic preorder on $B_0(\Sigma, K)$ iff there exists a nonempty subset $\mathbb{P}$ of $\Delta(\Sigma)$ such that:*

$$\varphi \succsim \phi \Longleftrightarrow \int \varphi \mathrm{d}p \geq \int \phi \mathrm{d}p \qquad \textit{for all } p \in \mathbb{P}. \tag{5}$$

Moreover, $\overline{co}^{w^*}(\mathbb{P})$ is the unique weak-closed and convex subset of $\Delta(\Sigma)$ representing $\succsim$ in the sense of above expression.

Given a functional $I : B_0(\Sigma) \to \mathbb{R}$, I is said to be:

- *monotonic* if $I(\varphi) \geq I(\phi)$, for all $\varphi, \phi \in B_0(\Sigma)$ such that $\varphi \geq \phi$;

- *constant additive* if $I(\varphi + a) = I(\varphi) + a$, for all $\varphi \in B_0(\Sigma)$ and $a \in \mathbb{R}$;
- *positively homogeneous* if $I(a\varphi) = aI(\varphi)$, for all $\varphi \in B_0(\Sigma)$ and $a \geq 0$;
- *constant linear* if it is constant additive and positively homogeneous.

A probability measure π is convex-ranged if, for every $0 < r < 1$ and every $A \in \mathscr{A}$, there is a subset $B \subset A$ with $B \in \mathscr{A}$ such that $\pi(B) = r\pi(A)$. Then a countably additive non-atomic measure is convex-ranged. Define now a function $\mu : \Sigma \to [0,1]$ such that, for any $E \in \Sigma : \mu(E) = \sup\{\pi(A) : A \subset E$ and $E \in \mathscr{A}\}$. Since π is countably additive, it is straightforward to show that the supremum can be reached. Call $A \in \mathscr{A}$ the *core* of E if $A \subseteq E$ and $\pi(A) = \mu(E)$ with A unique for a set of zero measure.

Let $A \in \mathscr{A}$, $M = \{1, \ldots, m\}$ and $\{B_i\}_{i \in M}$ be a finite partition of A. Let $\mathcal{M}$ be the set of all nonempty subsets of M and define, for $J \in \mathcal{M}$, $\mathcal{M}(J)$ as $\{K \in \mathcal{M} : K \subset J\}$. For $B^J = \cup_{j \in J} B_j$, let C^J be the core of B^J. Note that $C^M = A$. The *unanimous split* $\{\hat{E}^J\}_{J \in \mathcal{M}} \subset \mathscr{A}$ of $\{B_i\}_{i \in M}$ is inductively defined as follows: (1) for all $i \in M$, $\hat{E}^{\{i\}} = C^{\{i\}}$, and (2) for all J such that $|J| > 1$:

$$\widehat{E}^J := C^J \setminus \big(\bigcup_{K \in \mathcal{M}(J), K \neq J} \widehat{E}^K \big).$$

Note that $\{\hat{E}^J\}_{J \in \mathcal{M}}$ is a unanimous partition of A such that $\bigcup_{K \in \mathcal{M}(J)} \widehat{E}^J \subset B^J$, for all $J \in \mathcal{M}$, and $\mu(A^J) = \pi(C^J) = \sum_{K \in \mathcal{M}(J)} \mu(\widehat{E}^K)$. Moreover, for every simple act $f \in \mathcal{F}$ with range $\{x_1, \ldots, x_m\}$, let $\{\hat{E}^J(f)\}$ be the *ideal split* of $\{f^{-1}(x_i)\}$.

Lemma A2. *Let $f \in \mathcal{F}$ with range $\{x_1, \ldots, x_m\}$. Then, $(f_*, f^*) \in \mathcal{F}^2$ such that $(f_*(s), f^*(s)) = \big(\arg_{x_i} \min_{i \in J} u(x_i), \arg_{x_i} \max_{i \in J} u(x_i)\big)$, for $s \in \widehat{E}^J(f)$, is an envelope of f.*

Lemmas A1-2 are standard. Proofs are then omitted.

Lemma 1. *The collection $\mathscr{A}$ of unambiguous events is a λ-system.*

Proof of Lemma 1. The two first conditions of the definition of a λ-system are obviously satisfied. We only show the condition (iii) holds. Suppose a countable sequence of disjoint events $A_n \in \mathscr{A}$. Since $\mathscr{A}$ is a subset of the σ-algebra Σ, it is clear that the union of the events $A = \bigcup_n A_n$ belongs to Σ. By σ-additivity of every π_i, we have $\pi_i(A) = \sum_n \pi_i(A_n)$. For every A_n and i, j, $\pi_i(A_n) = \pi_j(A_n)$. Hence, if $\pi_i(A) = \pi_j(A)$, for every i, j, it implies that $A \in \mathscr{A}$. □

B APPENDIX — PROOF OF THEOREM 1

Theorem 1. HPC *holds if and only if social preferences $\succsim$ are represented by a consistently utilitarian RSEU.*

The proof of the necessity part is straightforward. We only demonstrate the sufficiency one.

Proof of Theorem 1. Assume HPC holds. We first show that, for all $A \in \mathscr{A}$, if $\pi_i(A) = p \in [0, 1]$, for all i, then $\pi(A) = p$, where $\pi(.)$ represents social beliefs.

Let $p_k = \frac{1}{2^k}$, where $k \in \mathbb{N}$. Take $A \in \mathscr{A}$ such that $\pi_i(A) = p_k$, for all i. We prove now, by induction, that $\pi(A) = p_k$. If $k = 1$, then $\pi_i(A) = \frac{1}{2}$ and $A \in \mathscr{A}$. We claim that $\pi(A) = \frac{1}{2}$. Suppose it is wrong, and wlog (*without loss of generality*) assume that $\pi(A) > \frac{1}{2}$. Therefore, there exist $x, y \in X$ such that $xAy \succ xA^c y$. However, for all i, $xAy \sim_i xA^c y$, which, by HPC, implies: $xAy \sim xA^c y$, *i.e.*, a contradiction. A similar argument works for the case where $\pi(A) < \frac{1}{2}$. Hence, $\pi_i(A) = \frac{1}{2}$, for all i, implies that $\pi(A) = \frac{1}{2}$. Now, suppose that $\pi_i(A) = \frac{1}{2^k}$, for all i, implies that $\pi(A) = \frac{1}{2^k}$. Assume that $\pi_i(A) = \frac{1}{2^{k+1}}$, for all i. We then want to show that $\pi(A) = \frac{1}{2^{k+1}}$. Suppose it is wrong and wlog assume that $\pi(A) > \frac{1}{2^{k+1}}$. By Lyapunov Theorem, there exists a subset $B \subset A^c$ such that $\pi_i(B) = \frac{1}{2^{k+1}}$, for all i. So, $B \in \mathscr{A}$ and $A \cap B = \emptyset$ imply that $A \cup B \in \mathscr{A}$. Since $\pi_i(A \cup B) = \frac{1}{2^k}$, for all i, by assumption, we have $\pi(A \cup B) = \frac{1}{2^k}$, which means $\pi(B) < \frac{1}{2^{k+1}}$. Similarly, there exists a subset $C \subset (A \cup B)^c$ s.t. $\pi_i(C) = \frac{1}{2^k}$, for all i. Hence, $\pi_i(A \cup C) = \frac{1}{2^k} = \pi_i(B \cup C)$, for all i, which implies that $\pi(A \cup C) = \frac{1}{2^k} = \pi(B \cup C)$. However, the first equality means that $\pi(C) < \frac{1}{2^{k+1}}$, while the second equality means that $\pi(C) > \frac{1}{2^{k+1}}$, that is a contradiction. The same argument works for the case where $\pi(A) < \frac{1}{2^{k+1}}$.

Now, first, take an arbitrary rational number $p \in (0, 1)$. Then p admits a finite dyadic expansion:

$$p = \sum_{k=1}^{m} \frac{x_k}{2^k},$$

where $x_k \in \{0, 1\}$. Take $A \in \mathscr{A}$ s.t. $\pi(A) = p$, for all i. Therefore, there exists a partition $\{A_1, \ldots, A_m\}$ of A s.t. $\pi_i(A_k) = \frac{x_k}{2^k}$, for all i and $k = 1, \ldots, m$. Thanks to the above analysis, we have $\pi(A_k) = \frac{x_k}{2^k}$, for all $k = 1, \ldots, m$. It is immediate to see that $\pi(A) = p$. Second, take an arbitrary irrational number $p \in (0, 1)$ and A s.t. $\pi_i(A) = p$, for all i. Suppose $\pi(A) \neq p$ and assume $\pi(A) > p$. There exists a rational number q s.t. $\pi(A) > q > p$. We can find an event B s.t. $A \subset B$ and $\pi_i(B) = q$, for all i. This requires that $\pi(B) = q < \pi(A)$, which contradicts the fact that $A \subset B$ implies $\pi(A) \leq \pi(B)$. A similar argument works for the case where $\pi(A) < p$. Hence, finally, $\pi(A) = p$.

We show now that the social utility u is a convex combination of individual utilities. Note that, for any nonnegative numbers $p_1, \ldots, p_m$ s.t. $\sum_{k=1}^{m} p_k = 1$, there exists a partition $\{A_k\}_{k=1}^{m}$ of S s.t. $\pi(A_k) = \pi_i(A_k) = p_k$, for all i and k. Therefore, for any vNM lottery L defined over X, we can construct an act $f \in \mathcal{L}$ s.t. the lottery L corresponds to the distribution on X generated by f. Conversely, any finitely valued act $f \in \mathcal{L}$ defines a distribution over X, which is a vNM lottery. In restricting preferences over $\mathcal{L}$, we can apply Harsanyi Theorem to conclude that u is a convex combination of $\{u_i\}_{i=1}^{n}$. □

C APPENDIX — PROOF OF THEOREM 2

Several notions and intermediate results are necessary.

A set function $\nu : \Sigma \to [0, 1]$ is a *capacity* if $\nu(\emptyset) = 0$, $\nu(S) = 1$ and $A \subseteq B$ implies $\nu(A) \leq \nu(B)$. Given π on $\mathscr{A}$, we define set functions $\mu_*, \mu^* : \Sigma \to [0, 1]$ by: for $A \in \Sigma$,

(6) $$\mu_*(A) = \sup_{\substack{B \subset A \\ B \in \mathscr{A}}} \{\pi(B)\} \quad \text{and} \quad \mu^*(A) = \inf_{\substack{A \subset B \\ B \in \mathscr{A}}} \{\pi(B)\}.$$

Lemma C1. *μ_* and μ^* are capacities on Σ .*

We omit the proof since it is straightforward.

Recall now the notion of Choquet integration. For any capacity ν and integrand $a : S \to \mathbb{R}$, the *Choquet* integral is defined by:

$$\oint a \mathrm{d}\nu = \int_0^{\infty} \nu(\{s : a(s) \geq t\})\mathrm{d}t + \int_{-\infty}^{0} [\nu(\{s : a(s) \geq t\}) - 1]\mathrm{d}t.$$

Therefore, a function $V : \mathcal{F} \to \mathbb{R}$ is said to be a *Choquet expected utility* (CEU) function if there exist a function u on X and a capacity μ on Σ s.t., for $f \in \mathcal{F}$:

$$V(f) = \oint u(f)\mathrm{d}\mu.$$

Lemma C2. *The conservative social opinion $\succsim_{cons}$ is represented by a function $V_c : \mathcal{F} \to \mathbb{R}$, where, for all $f \in \mathcal{F}$: $V_c(f) = \min_{p \in \mathbb{P}_\pi} u(f)\mathrm{d}p$. Similarly, the progressive social opinion $\succsim_{prog}$ is represented by $V_g : \mathcal{F} \to \mathbb{R}$, where, for all $f \in \mathcal{F}$: $V_g(f) = \max_{p \in \mathbb{P}_\pi} u(f)\mathrm{d}p$.*

Proof of Lemma C2. We will prove the result for V_c. The proof for V_g is analogous and, therefore, omitted.

First, show that for $f \in \mathcal{F}$: $V_c(f) = \sup\{\int u(g)\mathrm{d}\pi : u(f) \geq u(g)$ and $g \in \mathcal{L}\}$. By monotonicity, for all $g \in \mathcal{L}$, if $u(f) \geq u(g)$, then $V_c(f) \geq \int u(g)\mathrm{d}\pi$. Therefore, it is clear that $V_c(f) \geq \sup\{\int u(g)\mathrm{d}\pi : u(f) \geq u(g)$ and $g \in \mathcal{L}\}$. Now, suppose that $V_c(f) > \sup\{\int u(g)\mathrm{d}\pi : u(f) \geq u(g)$ and $g \in \mathcal{L}\}$. We want now to derive a contradiction. Since f is a simple act, there exist x, y in $\{x_1, \ldots, x_m\}$, which is the outcome set of the act f, s.t. $u(x) \geq u(z) \geq u(y)$, for all $z \in \{x_1, \ldots, x_m\}$. Therefore, $x \succsim f \succsim y$. Since $u(X)$ is convex, we know there exists $x_f \in X$ s.t. $x_f \sim f$, which implies $u(x) > \sup\{\int u(g)\mathrm{d}\pi : u(f) \geq u(g)$ and $g \in \mathcal{L}\}$. Again, by convexity of $u(X)$, there is a $y \in X$ such that: $u(x) > u(y) > \sup\{\int u(g)\mathrm{d}\pi : u(f) \geq u(g)$ and $g \in \mathcal{L}\}$. As a result, $f \succsim_c y$ while there is no $g \in \mathcal{L}$ s.t. $u(f) \geq u(g)$ and $g \succ_{cons} y$, which contradicts the definition of the conservative social opinion. Therefore, for $f \in \mathcal{F}$, $V_c(f) = \sup\{\int u(g)\mathrm{d}\pi : u(f) \geq u(g)$ and $g \in \mathcal{L}\}$.

Second, we want to show that, for $f \in \mathcal{F}$,

$$\oint u(f)\mathrm{d}\mu = \sup\{\int u(g)\mathrm{d}\pi : u(f) \geq u(g) \text{ and } g \in \mathcal{L}\}.$$

Notice that, according to Schmeidler (1989), CEU satisfies monotonicity, which means, if $u(f) \geq u(g)$, that $\oint u(f)\mathrm{d}\mu \geq \oint u(g)\mathrm{d}\mu$. When $g \in \mathcal{L}$, $\oint u(g)\mathrm{d}\mu = \int u(g)\mathrm{d}\pi$. Therefore, we have: $\oint u(f)\mathrm{d}\mu \geq \sup\{\int u(g)\mathrm{d}\pi : u(f) \geq u(g)$ and $g \in \mathcal{L}\}$. Take an act $f \in \mathcal{F}$. Wlog, we can write $f = x_1 A_1 x_2 A_2 \cdots x_m A_m$, where $u(x_1) > u(x_2) > \cdots > u(x_m)$. So,

$$\oint u(f)\mathrm{d}\mu = \sum_{k=1}^{m-1} [u(x_k) - u(x_{k+1})]\mu(\cup_{j=1}^{k} A_j) + u(x_m).$$

Let $\widehat{E}^J(f)$ be the ideal split of $\{f^{-1}(x_i)\}$. Consider $g \in \mathcal{L}$ defined by, for all $s \in S$, $g(s) = \arg\min_{i \in J} u(x_i)$ if $s \in \widehat{E}^J(f)$. We want to show that:

$$\int u(g)\mathrm{d}\pi = \sup\{\int u(g)\mathrm{d}\pi : u(f) \geq u(g) \text{ and } g \in \mathcal{L}\}.$$

For $1 \leq k \leq m$, we write $\widehat{E}^J_{\leq k}(f) := \bigcup_J \{\widehat{E}^J(f) : J \subseteq \{1, 2, \ldots, k\}$ and $k \in J\}$. Since $x_1 \succ x_2 \succ \ldots \succ x_m$, we can rewrite g in the following way: for all $s \in S$, $g(s) = x_k$ if $s \in \widehat{E}^J_{\leq k}(f)$. We know that, for every $1 \leq k \leq m$, $\sum_{j=1}^{k} \pi\big(\widehat{E}^J_{\leq j}(f)\big) = \pi(C^{\{1,\ldots,k\}})$, which implies $\pi(\widehat{E}^J_{\leq k}(f)) = \pi(C^{\{1,\ldots,k\}}) - \pi(C^{\{1,\ldots,k-1\}})$. Therefore:

$$\begin{aligned}
V_c(g) &= \sum_{k=1}^{m} u(x_k)\pi(\widehat{E}^J_{\leq k}(f)) \\
&= \sum_{k=1}^{m} u(x_k)[\pi(C^{1,\ldots,k}) - \pi(C^{1,\ldots,k-1})] \\
&= \sum_{k=1}^{m} u(x_k)[\mu_*(\cup_{j=1}^{k} A_j) - \mu_*(\cup_{j=1}^{k-1} A_j)] \\
&= \oint u(f)\mathrm{d}\mu.
\end{aligned}$$

Hence, there exists $p_* \in \mathbb{P}_\pi$ s.t. $p_*(A_k) = \mu_*(\cup_{j=1}^{k} A_j) - \mu_*(\cup_{j=1}^{k-1} A_j)$. That is, $\oint u(f)\mathrm{d}\mu = \int u(f)\mathrm{d}p_* \geq \min_{p \in \mathbb{P}_\pi} u(f)\mathrm{d}p$. However, $u(f) \geq u(g)$ implies that, for all $p \in \mathbb{P}_\pi$, $\int u(f)\mathrm{d}p \geq \int u(g)\mathrm{d}p = \int u(g)\mathrm{d}\pi$. So, $\min_{p \in \mathbb{P}_\pi} u(f)\mathrm{d}p \geq \int u(g)\mathrm{d}\pi = \oint u(f)\mathrm{d}\mu$. Hence, we have $\int u(f)\mathrm{d}\mu = \min_{p \in \mathbb{P}_\pi} u(f)\mathrm{d}p$. □

Recall now that $\overline{u}^f_{\mathbb{P}} \equiv \max_{p\in\mathbb{P}} \int u(f)\mathrm{d}p$ and $\underline{u}^f_{\mathbb{P}} \equiv \min_{p\in\mathbb{P}} \int u(f)\mathrm{d}p$.

Theorem 2. *Suppose* $\Theta = \{cons, prog\}$. Unanimity *holds if and only if social preferences* $\succsim$ *are represented by a consistently utilitarian GHEU.*

The proof of the necessity part is straightforward. We only prove the sufficiency one.

Proof of Theorem 2. From previous analysis, we know that, for $f, g \in \mathcal{F}$, $f \succsim_{cons} g$ iff $\underline{u}^f_{\mathbb{P}_\pi} \geq \underline{u}^g_{\mathbb{P}_\pi}$ and $f \succsim_{prog} g$ iff $\overline{u}^f_{\mathbb{P}_\pi} \geq \overline{u}^g_{\mathbb{P}_\pi}$. Thus, Unanimity implies $f \succsim g$, whenever $\underline{u}^f_{\mathbb{P}_\pi} \geq \underline{u}^g_{\mathbb{P}_\pi}$ and $f \succsim_{prog} g$ iff $\overline{u}^f_{\mathbb{P}_\pi} \geq \overline{u}^g_{\mathbb{P}_\pi}$. Therefore, there exists a monotonic function $W : u(X) \times u(X) \to \mathbb{R}$ s.t. $W(\underline{u}^f_{\mathbb{P}_\pi}, \underline{u}^f_{\mathbb{P}_\pi}) = W(\underline{u}^g_{\mathbb{P}_\pi}, \overline{u}^g_{\mathbb{P}_\pi})$, whenever $\underline{u}^f_{\mathbb{P}_\pi} = \underline{u}^g_{\mathbb{P}_\pi}$ and $\underline{u}^f_{\mathbb{P}_\pi} = \overline{u}^g_{\mathbb{P}_\pi}$. Thus, for any act f, the associated pair $(\underline{u}^f_{\mathbb{P}_\pi}, \underline{u}^f_{\mathbb{P}_\pi})$ characterizes the indifference class with respect to f. Hence, W is a representation of $\succsim$ on $\mathcal{F}$. □

D Appendix — Proof of Theorem 3

Theorem 3. *Suppose* $\Theta = \{cons, prog\}$. Unanimity *and* Independence *hold if and only if social preferences* $\succsim$ *are represented by a consistently utilitarian HEU.*

The necessity part is straightforward and, therefore, omitted. Then, we just show the sufficiency part. The proof consists of demonstrating, step by step, three intermediate lemmas, *i.e.*, Lemma D1-3. Observe first that $\{u(f) : f \in \mathcal{F}\} = \{\phi \in B_0(\Sigma) : \phi = u(f)$, for some $f \in \mathcal{F}\} = B_0(\Sigma, u(X))$. Wlog, assume that $[-1, 1] \subset u(X)$. Define I on $B_0(\Sigma, u(X))$ a follows: for all $f \in \mathcal{F}$, $I(u(f)) = V(f)$. Note that $f \succsim g$ iff $I(u(f)) \geq I(u(g))$, for all $f, g \in \mathcal{F}$. Moreover, $I(\mathbb{1}) = 1$.

Lemma D1. *I is positively homogeneous.*

Proof of Lemma D1. For $\varphi \in B_0(\Sigma, u(X))$ and $a \geq 0$, show that $I(a \cdot \varphi) = aI(\varphi)$. Let $f \in \mathcal{F}$ be an act s.t. $I(\varphi) = V(f)$. Let $x_0 \in X$ be defined by $u(x_0) = 0$. By continuity and monotonicity, there exists $x \in X$ s.t. $u(x) = V(f)$. Consider $a \in (0, 1)$. Thanks to the convexity of π, there exists a subset $A \in \mathscr{A}$ s.t. $\pi(A) = a$. Let $g \in \mathcal{F}$ be defined as follows: for all $s \in S$, $g(s) \sim f(s)Ax_0$. Since $u(g(s)) =$

$a \cdot u(f(s))$, for all s, we have: $V(g) = I(a\varphi)$. Furthermore, note that f and g admit the same ideal splitting. Therefore, $V^*(g) = V^*(f[A]x_0)$ and $V_*(g) = V_*(f[A]x_0)$, that is:

$$\oint u(g)\mathrm{d}\mu^* = a \cdot \oint u(f)\mathrm{d}\mu^* \quad \text{and} \quad \oint u(g)\mathrm{d}\mu_* = a \cdot \oint u(f)\mathrm{d}\mu_*.$$

Hence, if $f \sim^* x^*$ and $f \sim_* x_*$, then $g \sim^* x^*Ax_0$ and $g \sim_* x_*Ax_0$. By Independence, we have $g \sim xAx_0$, which means $V(g) = a \cdot u(x) = a \cdot V(f)$. Hence, $I(a\varphi) = aI(\varphi)$, for $a \in (0,1)$. If $a = 0$ or $a = 1$, the result holds trivially. If $a > 1$, then $\frac{1}{a}I(a \cdot \varphi) = I(\varphi)$ according to the above argument. This ends the proof. □

It is now sufficient to extend I by homogeneity to all $B_0(\Sigma)$. Note that I is monotone and positively homogeneous on $B_0(\Sigma)$.

Lemma D2. *I is constant additive.*

Proof of Lemma D2. Let $\varphi \in B_0(\Sigma)$ and $a \in \mathbb{R}$. We want to show $I(\varphi + a \cdot \mathbb{1}) = I(\varphi) + a$. Let $f \in \mathcal{F}$ be s.t. $u(f) = 2\varphi$ and $x \in X$ be s.t. $u(x) = 2a$. Also, by continuity and monotonicity, there is $y \in X$ s.t. $f \sim y$. By convexity of π, take $A \in \mathscr{A}$ s.t. $\pi(A) = \frac{1}{2}$. Define act $g \in \mathcal{F}$ by for all s, $g(s) \sim f(s)Ax$. So, $u(g(s)) = \frac{u(f(s))+u(x)}{2}$ for all s, which implies $u(g) = \varphi + a \cdot \mathbb{1}$. Since f and g have identical ideal splitting, we must have $g \sim^* f[A]x$ and $g \sim_* f[A]x$. Therefore,

$$\oint u(g)\mathrm{d}\mu^* = \frac{1}{2}\Big[\oint u(f)\mathrm{d}\mu^* + u(x)\Big] \quad \text{and} \quad \oint u(g)\mathrm{d}\mu_* = \frac{1}{2}\Big[\oint u(f)\mathrm{d}\mu_* + u(x)\Big].$$

Let $y \sim^* f^*$ and $y \sim_* f_*$. Then, $g \sim^* y^*Ax$ and $g \sim_* y_*Ax$. According to Independence, we have $g \sim yAx$. Therefore, $I(\varphi + a \cdot \mathbb{1}) = \frac{1}{2}(u(y) + u(x)) = \varphi + a$. □

Let $B_0(\mathscr{A})$ denote the set of all real-valued $\mathscr{A}$-measurable finite valued functions. For $\varphi \in B_0(\Sigma)$, let

$$\varphi^* = \arg \inf_{\substack{\phi \in B_0(\mathscr{A}) \\ \phi \geq \varphi}} I(\phi) \quad \text{and} \quad \varphi_* = \arg \sup_{\substack{\phi \in B_0(\mathscr{A}) \\ \varphi \geq \phi}} I(\phi)$$

Note that, for $f \in \mathcal{F}$, $V_g(f) = I(u(f)^*)$ and $V_c(f) = I(u(f)_*)$.

Lemma D3. *Let $I : B_0(\Sigma) \to \mathbb{R}$ be a monotonic constant linear functional. Then, there exists unique $\alpha \in [0,1]$ such that, for all $\varphi \in B_0(\Sigma)$, $I(\varphi) = \alpha I(\varphi^*) + (1 - \alpha)I(\varphi_*)$.*

Proof of Lemma D3. By Theorem 2, we know that $I(\varphi) = W(I(\varphi_*), I(\varphi^*))$. Since I is homogeneous and constant additive, we have, for $\alpha \in [0,1]$ and $a \in \mathbb{R}$:

$$W(\alpha\varphi_*, \alpha\varphi^*) = \alpha W(\varphi_*, \varphi^*),$$
$$W(\varphi_* + a \cdot \mathbb{1}, \varphi^* + a \cdot \mathbb{1}) = W(\varphi_*, \varphi^*) + a.$$

Note that $I(\varphi - I(\varphi_*)) = W(I(\varphi_*) - I(\varphi_*), I(\varphi^*) - I(\varphi_*)) = W(0, I(\varphi^*) - I(\varphi_*))$. Therefore:

$$\begin{aligned} W(I(\varphi_*), I(\varphi^*)) &= W(0, I(\varphi^*) - I(\varphi_*)) + I(\varphi_*) \\ &= W(0,1)(I(\varphi^*) - I(\varphi_*)) + I(\varphi_*). \end{aligned}$$

Let $\alpha = W(0,1)$. We have: $I(\varphi) = \alpha I(\varphi^*) + (1 - \alpha)I(\varphi_*)$. Let also φ be s.t. $I(\varphi_*) = 0$. Then, monotonicity implies that $I(\varphi) = \alpha > 0$. □

This ends the proof of Theorem 3.

E Appendix — Proof of Theorem 4 and 5

The proofs of Theorem 4 and 5 proceed in six steps, corresponding to six lemmas, *i.e.*, Lemma E1-6. Lemma E1-3, which do not assume Independence, are sufficient for the proof of Theorem 4. Lemma 4-6, which assume Independence, are used to derive Theorem 5.

The necessity of both theorems are standard, we hence omit it.

Lemma E1. *There exists a unique non-empty convex and compact set $\mathbb{P} \subseteq \mathbb{P}_\pi$ of probabilities on Σ such that, for all $f, g \in \mathcal{F}$:*

$$f \trianglerighteq g \iff \int u(f)\mathrm{d}p \geq \int u(g)\mathrm{d}p \quad \textit{for all } p \in \mathbb{P}.$$

Proof of Lemma E1. By definition of the binary relation $\trianglerighteq$, we know, for all $h \in \mathcal{F}$ and $A \in \mathscr{A}$, that:

$$
\begin{aligned}
f \trianglerighteq g &\Longleftrightarrow I(u(f[A]h)) \geq I(u(g[A]h)) \\
&\Longleftrightarrow I(\pi(A)u(f) + (1-\pi(A))u(h)) \geq I(\pi(A)u(g) + (1-\pi(A))u(h))
\end{aligned}
$$

Since π has a convex range on $\mathscr{A}$, for each $\lambda \in [0,1]$, there exists $A \in \mathscr{A}$ s.t. $\lambda = \pi(A)$. Therefore, for all $\lambda \in (0,1)$ and $h \in \mathcal{F}$: $f \trianglerighteq g \Longleftrightarrow I(\lambda u(f) + (1-\lambda)u(h)) \geq I(\lambda u(g) + (1-\lambda)u(h))$. Now, we define $\succeq$ on $B_0(\Sigma, u(X))$ as follows: for all $\varphi, \phi \in B_0(\Sigma, u(X))$: $\varphi \succeq \phi \Longleftrightarrow I(\lambda\varphi + (1-\lambda)\psi) \geq I(\lambda\phi + (1-\lambda)\psi), \forall \psi \in B_0(\Sigma, u(X)), \lambda \in (0,1]$. Hence, it is straightforward that $f \trianglerighteq g \Leftrightarrow u(f) \succeq u(g)$. Therefore, $\succeq$ is obviously a non-trivial, monotonic and conic preorder on $B_0(\Sigma, u(X))$. According to Bewley (2002) or Ghirardato, Maccheroni and Marinacci (2004), we know that there exists a unique non-empty convex and compact set $\mathbb{P}$ of probabilities on Σ s.t., for all $\varphi, \phi \in B_0(\Sigma, u(X))$: $\varphi \succeq \phi \Longleftrightarrow \int \varphi \mathrm{d}p \geq \int \phi \mathrm{d}p$, for all $p \in \mathbb{P}$. We are left to show that $\mathbb{P} \subseteq \mathbb{P}_\pi$. Suppose it is wrong, *i.e.*, there exists $p \in \mathbb{P}$ s.t. $p \notin \mathbb{P}_\pi$. Since $\mathbb{P}_\pi$ contains all extensions of π, p is not an extension of π. So there exists $A \in \mathscr{A}$ s.t. $p(A) \neq \pi(A)$. Wlog, assume $p(A) \geq \pi(A)$. Then, for $u(x) > u(y)$, we have: $u(x)P(A) + u(y)(1-P(A)) > u(x)\pi(A) + u(y)(1-\pi(A))$. By continuity of u, there exists $z \in X$ such that: $u(x)P(A) + u(y)(1-P(A)) > u(z) > u(x)\pi(A) + u(y)(1-\pi(A))$. Therefore, $z \trianglerighteq_\theta xAy$, for all $\theta \in \{p, o\}$ and $z \ntrianglerighteq xAy$, which contradicts Unanimity. □

Lemma E2. *For each* $\varphi \in B_0(\Sigma, u(X))$*, we have:* $\min_{p\in\mathbb{P}} \int \varphi \mathrm{d}p \leq I(\varphi) \leq \max_{p\in\mathbb{P}} \int \varphi \mathrm{d}p$.

Proof of Lemma E2. The proof is made by negation. First, suppose that there exists $\varphi \in B_0(\Sigma, u(X))$ s.t. $I(\varphi) < \min_{p\in\mathbb{P}} \int \varphi \mathrm{d}p$. Let $f \in \mathcal{F}$ and $x \in X$ be s.t. $u(f) = \varphi$ and $u(x) = \phi$. Suppose that $f \sim x$. Then, $I(\varphi) = I(\phi)$. However, for all $\lambda \in (0,1]$ and $\psi \in B_0(\Sigma, u(X))$, we have: $I(\lambda\varphi + (1-\lambda)\psi) > I(\lambda\phi + (1-\lambda)\psi)$, which implies that $f \triangleright x$. This contradicts the assumption whereby $f \sim x$. A similar argument works when $I(\varphi) > \max_{p\in\mathbb{P}} \int \varphi \mathrm{d}p$. □

Lemma E3. *The optimistic social opinion* $\succsim_o$ *is represented by* V_o *s.t., for* $f \in \mathcal{F}$*,*

$V_o(f) = \max_{p\in\mathbb{P}} \int u(f)\mathrm{d}p$. *Moreover, the pessimistic social opinion* $\succsim_p$ *is represented by* V_p *s.t., for* $f \in \mathcal{F}$, $V_p(f) = \min_{p\in\mathbb{P}} \int u(f)\mathrm{d}p$.

Proof of Lemma E3. Prove the result for V_o. The proof for V_g is analogous and, therefore, omitted. Note that $\succsim_o$ admits a restricted SEU representation. So, by monotonicity and continuity, for each act $f \in \mathcal{F}$, there exists $x_f \in X$ s.t. $x_f \sim_o f$. If $x_f \not\trianglerighteq f$, then there exist $g \in \mathcal{F}$ and $A \in \mathscr{A}$ s.t. $f[A]g \succ x_f[A]g$. By definition of $\succsim_o$, we have $f \succ_o x_f$. Hence, $x_f \trianglerighteq f$. This implies that $u(x_f) \geq \int u(f)\mathrm{d}p$, for all $p \in \mathbb{P}$, which means $u(x_f) \geq \max_{p\in\mathbb{P}} \int u(f)\mathrm{d}p$. Now, suppose that, for some $f \in \mathcal{F}$, $u(x_f) > \max_{P\in\mathbb{P}} \int u(f)\mathrm{d}P$. Take x_{min} be s.t. $u(f(s)) \geq u(x_{min})$, for all s. Then, there exists an event $A \in \mathscr{A}$ such that: $u(x_f) > \max_{P\in\mathbb{P}} \int u(f)\mathrm{d}P = u(x_f A x_{min})$. Let $z \in X$ be s.t. $z \sim x_f A x_{min}$. We have: $x_f \succ_o z \trianglerighteq f$, which then implies $x_f \succ_o f$, that is a contradiction. In conclusion, $V_o(f) = \max_{p\in\mathbb{P}} \int u(f)\mathrm{d}p$, represents $\succsim_o$. □

Theorem 4. *Suppose* $\Theta = \{p, o\}$. Unanimity *holds if and only if social preferences* $\succsim$ *are represented by a consistently utilitarian GMEU with* $\mathbb{P} \subseteq \mathbb{P}_\pi$.

Proof of Theorem 4. As seen previously, we know that, for $f, g \in \mathcal{F}$, $f \succsim_p g$ iff $\underline{u}_{\mathbb{P}}^f \geq \underline{u}_{\mathbb{P}}^g$ and $f \succsim_o g$ iff $\overline{u}_{\mathbb{P}}^f \geq \overline{u}_{\mathbb{P}}^g$. Thus, Unanimity implies $f \succsim g$, whenever $\underline{u}_{\mathbb{P}}^f \geq \underline{u}_{\mathbb{P}}^g$ and $f \succsim_o g$ iff $\overline{u}_{\mathbb{P}}^f \geq \overline{u}_{\mathbb{P}}^g$. Therefore, there exists a monotonic function $W : u(X) \times u(X) \to \mathbb{R}$ s.t. $W(\underline{u}_{\mathbb{P}}^f, \underline{u}_{\mathbb{P}}^f) = W(\underline{u}_{\mathbb{P}}^g, \overline{u}_{\mathbb{P}}^g)$, whenever $\underline{u}_{\mathbb{P}}^f = \underline{u}_{\mathbb{P}}^g$ and $\underline{u}_{\mathbb{P}}^f = \overline{u}_{\mathbb{P}}^g$. Thus, for any act f, the associated pair $(\underline{u}_{\mathbb{P}}^f, \underline{u}_{\mathbb{P}}^f)$ characterizes the indifference class with respect to f. Hence, W is a representation of $\succsim$ on $\mathcal{F}$. □

Theorem 5. *Suppose* $\Theta = \{p, o\}$. Unanimity *and* Independence *hold if and only if social preferences* $\succsim$ *are represented by a consistently utilitarian* α*-MEU with* $\mathbb{P} \subseteq \mathbb{P}_\pi$.

The proof is based on the three following lemmas:

Lemma E4. *I is positively homogeneous.*

Proof of Lemma E4. For $\varphi \in B_0(\Sigma, u(X))$ and $a \geq 0$, show that $I(a \cdot \varphi) = aI(\varphi)$. Let $f \in \mathcal{F}$ be s.t. $I(\varphi) = V(f)$ and let $x_0 \in X$ be s.t. $u(x_0) = 0$. By continuity and monotonicity, there exists $x \in X$ s.t. $u(x) = V(f)$. Take now $a \in (0, 1)$.

Because of the convexity of π, there exists $A \in \mathscr{A}$ s.t. $\pi(A) = a$. Let an act $g \in \mathcal{F}$ be defined by, for all $s \in S$, $g(s) \sim f(s)Ax_0$. Since $u(g(s)) = a \cdot u(f(s))$, for all s, we have $V(g) = I(a\varphi)$. Furthermore, note that V_o and V_p satisfy homogeneity and constant additivity. Therefore, $V_o(g) = aV_o(f)$ and $V_p(g) = aV_p(f)$. Hence, let $x^*, x_* \in X$ be s.t. $f \sim_o x^*$ and $f \sim_p x_*$. Then, $g \sim_o x^*Ax_0$ and $g \sim_p x_*Ax_0$. By Independence, we have $g \sim xAx_0$, which leads to $V(g) = a \cdot u(x) = a \cdot V(f)$. Thus, $I(a\varphi) = aI(\varphi)$, for $a \in (0, 1)$. If $a = 0$ or $a = 1$, the result holds trivially. If $a > 1$, then $\frac{1}{a}I(a \cdot \varphi) = I(\varphi)$ according to the above argument. This ends the proof. $\square$

By homogeneity, we now extend I to all $B_0(\Sigma)$. Note that I is monotone and positively homogeneous on $B_0(\Sigma)$.

Lemma E5. *I is constant additive.*

Proof of Lemma E5. Let $\varphi \in B_0(\Sigma)$ and $a \in \mathbb{R}$. We want to show that $I(\varphi + a \cdot \mathbb{1}) = I(\varphi) + a$. Let $f \in \mathcal{F}$ be s.t. $u(f) = 2\varphi$ and $x \in X$ be s.t. $u(x) = 2a$. In addition, by continuity and monotonicity, there is $y \in X$ s.t. $f \sim y$. By the convexity of π, take $A \in \mathscr{A}$ s.t. $\pi(A) = \frac{1}{2}$. Define act $g \in \mathcal{F}$ by, for all s, $g(s) \sim f(s)Ax$. Hence, $u(g(s)) = \frac{u(f(s))+u(x)}{2}$, for all s, which implies $u(g) = \varphi + a \cdot \mathbb{1}$. Since V_o and V_p are constant additive, we have $V_o(g) = \frac{1}{2}V_o(f) + a$ and $V_p(g) = \frac{1}{2}V_p(f) + a$. Let $y^*, y_* \in X$ be s.t. $f \sim_o y^*$ and $f \sim_p y_*$. Then, $g \sim_o y^*Ax$ and $g \sim_p y_*Ax$. By Independence, we have $g \sim yAx$. Therefore, $I(\varphi + a \cdot \mathbb{1}) = \frac{1}{2}(u(y) + u(x)) = \varphi + a$. $\square$

Lemma E6. *There exists a unique $\alpha \in [0, 1]$ s.t. $I(\varphi) = \alpha I^*(\varphi) + (1 - \alpha)I_*(\varphi)$.*

Proof of Lemma E6. Since I satisfies homogeneity and constant additivity, according to Lemma E5, there exists a unique $\alpha \in [0, 1]$ s.t. the above expression holds. $\square$

This ends the proof of Theorem 5.

F Appendix — Proof of Theorem 6

Theorem 6. *Suppose* $\Theta = \{B, m\}$. Unanimity *and* Independence *hold if and only if social preferences* $\succsim$ *are represented by a consistently utilitarian cMEU with* $\mathbb{P} = \mathbb{P}_\pi$ *and* p *is a convex combination of* $\{\pi_i\}_{i=1}^n$.

Since the necessity part is straightforward, we only prove the sufficiency one.

Proof of Theorem 6. For all acts $f, g \in \mathcal{F}$, $f \succsim_B g$ iff $u_{p^*}^f \geq u_{p^*}^g$ and $f \succsim_m g$ iff $\alpha \overline{u}_{\mathbb{P}_\pi}^f + (1-\alpha)\underline{u}_{\mathbb{P}_\pi}^f \geq \alpha \overline{u}_{\mathbb{P}_\pi}^g + (1-\alpha)\underline{u}_{\mathbb{P}_\pi}^g$. Unanimity implies the existence of $W : u(X) \times u(X) \to \mathbb{R}$ s.t.: $W(u_{p^*}^f, \alpha \overline{u}_{\mathbb{P}_\pi}^f + (1-\alpha)\underline{u}_{\mathbb{P}_\pi}^f) \geq W(u_{p^*}^g, \alpha \overline{u}_{\mathbb{P}_\pi}^g + (1-\alpha)\underline{u}_{\mathbb{P}_\pi}^g)$, whenever $u_{p^*}^f \geq u_{p^*}^g$ and $\alpha \overline{u}_{\mathbb{P}_\pi}^f + (1-\alpha)\underline{u}_{\mathbb{P}_\pi}^f \geq \alpha \overline{u}_{\mathbb{P}_\pi}^g + (1-\alpha)\underline{u}_{\mathbb{P}_\pi}^g)$. Hence, $V(f) = W(u_{p^*}^f, \alpha \overline{u}_{\mathbb{P}_\pi}^f + (1-\alpha)\underline{u}_{\mathbb{P}_\pi}^f)$ represents social preferences $\succsim$. Since $\succsim_B$ and $\succsim_m$ are constantly independent, a similar argument can be used as in Lemmas E4 and E5 and Independence implies that I defined as $I(u(f)) = V(f)$ is homogeneous and constantly additive. Therefore, Lemma D3 yields the existence of a unique $\varepsilon \in [0, 1]$ s.t., for $f \in \mathcal{F}$, $V(f) = \varepsilon u_{p^*}^f + (1-\varepsilon)(\alpha \overline{u}_{\mathbb{P}_\pi}^f + (1-\alpha)\underline{u}_{\mathbb{P}_\pi}^f)$. □

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